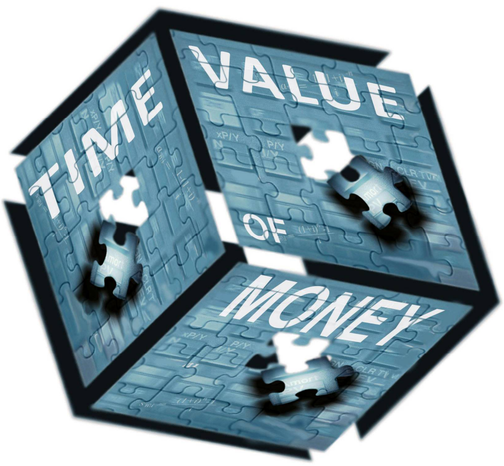

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BRIEF NOTES ON TIME VALUE OF MONEY

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A few words about me and about these notes

I'm a Portuguese teacher. For almost 30 years, I've been teaching at Escola Superior de Tecnologia e Gestão, one of the five Schools that make part of Instituto Politécnico de Viseu.

My favorite course is *Cálculo Financeiro* (Portuguese name), which relies on Time Value of Money and its applications to real life situations.

Some years ago I started receiving Erasmus students from many European countries (mostly from Germany, Greece, Lithuania, Poland, Spain, Turkey) who enrolled in my *Time Value of Money* course, specifically prepared for Erasmus.

I wrote some academic books about *Cálculo Financeiro*, luckily adopted in many Universities, not only in Portugal, but also in other Portuguese speaking countries like Angola, Mozambique, Cabo Verde and Brazil.

This is something I love to do: writing academic material. So, I decided to prepare and deliver some notes in English that could help my Erasmus students attending *Time Value of Money*.

Well, as time passed by, some of them contacted me saying that these simple notes were useful for them later, either in their studies or even in their jobs.

That made me think that maybe they could be useful for other students, also.

That's why I decided to deliver them for free, using a specific webpage (www.time-value-of-money.com).

Anyone can download these notes. However, I would really appreciate if you could leave a brief testimonial on my webpage, either you like it or not. Your opinion, your suggestion, your correction are really meaningful and very important for me.

One last note: along the text, I will use both Euro and USD as currency. Accordingly, I will use European and American notation for numbers, i.e., for instance, €1.234,56 or $1,234.56, as well as 0,05 or 0.05, respectively.

I sincerely hope you find these notes useful.

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1. INTRODUCTION

1.1 - Time value of money.

Imagine that you won a prize, say €1.000, and you are asked *Do you prefer to receive this prize today or one year from now?* Almost for sure you would prefer to receive it today.

In fact, we value differently the same amount of money according to the moment it is available for us. We tend to value more a given amount of money the sooner we have it. Why? Well, we can find some good reasons. For instance, if we have the money today,

1. We can buy things that we need or that just make us happy;
2. We can buy things today that may be more expensive one year from now (inflation);
3. We can make a deposit on a bank and receive a higher amount one year later;
4. We can spend part of that amount and deposit the rest;
5. We eliminate the risk of, for some reason, not receiving that money one year from now. The later we receive, the higher the risk associated (it's safer to receive now than the promise of receiving later).

This shows the importance of **time** in any situation involving money. This is usually known as **"time value of money"**, which is the key concept concerning any situation involving money.

Every time we need to *deal* with different amounts of money (for instance, compare them or sum them) we must have this in mind: €1.000 today is different from €1.000 one year from now (or any other moment in the future or in the past). It's as if those amounts are different *units*; so, they are not directly comparable; we can't just sum or subtract them *directly*. If, for any reason, we have to operate with them, we must first make them directly comparable, it means, express them on *one same unit*. For example, trying to sum them directly would be something like trying to sum directly meters and kilometers (different units). That would be a tremendous error. Of course, first of all, we must express every amount on the same unit (for instance, meters or kilometers or any other, but all of them on that same unit).

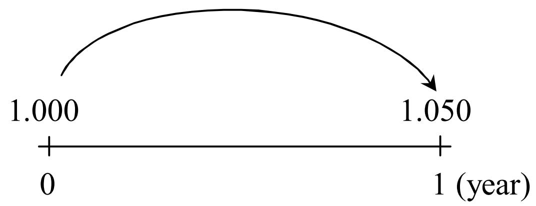
When dealing with amounts of money (let's call them *cash-flows*) the idea is the same. We must express all of them on one same *unit*. And what determines the unit, when dealing with money? **Time**! So, we must express every single cash-flow at **one same moment**. Only after every cash-flow is expressed at one same moment we can compare them, sum them or subtract them, for instance.

> **Golden Rule of Financial Calculus**: in order to correctly compare or operate with cash-flows, all of them must be expressed at one same moment.

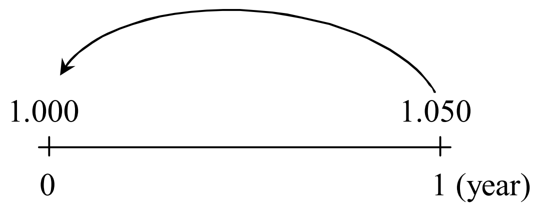
How do we do this? That's what we will see later. For now, we can only say that we may need to express one or more cash-flows at a <u>later moment</u> (later than the moment it is expressed or it really occurs) and/or one or more cash-flows at an <u>earlier moment</u> (earlier than the moment it is expressed or it really occurs). The techniques we can use in order to achieve this will be studied soon.

It should be understandable that a cash-flow expressed at a later moment will be higher (for instance €1.000 today may be worth, let's say, €1.050 one year from today, i.e., later). Following the same way of thinking, a cash-flow expressed at an earlier moment will be smaller (for instance, €1.050 one year from now, may be worth, let's say, €1.000 today, i.e., earlier).

We usually feel more comfortable thinking on situations of the first type. We usually understand them more easily. For instance, if we deposit €1.000 today on a bank, we hope to have more money one year from now, let's say, €1.050.



We usually feel less comfortable thinking on situations of the second type. But imagine this: some time ago you borrowed a loan and so you must pay €1.050 one year from now. Suppose that you intend to pay your debt today (earlier than what was agreed and supposed to). Should you pay (today) €1.050? Is this fair? Maybe you agree that paying *sooner* you should pay *less* than €1.050, let's say, €1.000, right?



See? This is time value of money…

1.2 - Key variables: money, time, interest rate.

There are three key variables when dealing with cash-flows. Actually, we already mentioned them. They are **money**, **time** and **interest** (or, as we will see, **interest rate**).

Above we have already directly referred to the first two key variables: money and time. In an indirect way, we have also mentioned the third one, interest rate. When we said that *€1.000 today may be worth, let's say, €1.050 one year later* what we were really saying was that the value of time (in this case, the value of one year) was €50, i.e., 50/1.000 or 5% (per year). This is the *interest rate*, which is usually expressed as *percent per year* (although this is not mandatory). So, we can say that the interest rate (% per year) is the price of holding 100 euro for one year. These €50 are the *interest.*

1.3 - Interest: why and how.

Why is there interest? Well, it seems fair that if we borrow some amount of money from someone for a given period of time, we must give him/her back that amount *plus something*.

In fact, we used that money (not *our* money, indeed) for that period. That has a price. That price is the interest. How can it be computed? Well, it is easy to understand that the amount of interest (in euro) will depend on *how much* money we borrowed, *for how long* and *its price* (i.e., the *interest rate* – let's say, 5% per year).

So, for example, if we borrow €5.000 for 1 year and the interest rate is 5% per year (or annual, as it is common to say), we must pay at the end of the year, not only the €5.000, but also

5.000 x 1 x 0,05 = 250 euro

This is the interest.

Let's adopt some symbols. Be

. **I** the interest (in Euro, USD or any other currency)
. $\mathbf{C_0}$ the amount of money at the beginning of period (say, moment 0), also called *initial capital, proceeds, principal, **P*** or *present value,* **PV**)
. **n** the time (expressed in years, quarters, months, days, whatever)
. **i** the interest rate (usually expressed in % per year)

So, we can say that

$$\mathbf{I = C_0 \,.\, n \,.\, i} \quad \text{or} \quad \mathbf{I = PV \,.\, n \,.\, i}$$

It is mandatory that **n** and **i** are both expressed on the same unit. For instance, **n** in years and **i** per year, or **n** in months and **i** per month. When units diverge, one must be converted. Let's go through a few simple examples.

Example 1

C_0 (or PV) = \$1,000; n = 2 years; i = 4% (annual); I = ?

I = 1,000 x 2 x 0.04 = \$80

Example 2

C_0 (or PV) = \$1,000; n = 5 months; i = 4% (annual); I = ?

I = 1 000 x 5/12 x 0.04 = \$16.67

Example 3

C_0 (or PV) = \$1,000; n = 123 days; i = 4% (annual); I = ?

I = 1 000 x 123/365 x 0.04 = \$13.48

In the next chapter we will see that there are several ways to compute days between dates.

2. SIMPLE INTEREST AND COMPOUND INTEREST

2.1 - Simple interest. Interest rates under simple interest.

Simple interest is characterized by the fact that interest produced in any period will not earn interest in the forthcoming periods, i.e., there is no interest on interest. Interest of any period is always computed only upon C_0 (PV), so it will remain constant from period to period.

Example 4

C_0 (PV) = \$1,000; n = 3 years; i = 12% (annual)

I_1 = 1,000 x 1 x 0,12 = \$120

After 1 year, the interest is \$120, but this amount will not be added to the initial \$1,000. On the second year, the amount that will produce interest remains \$1,000 (C_0 or PV). So,

I_2 = 1,000 x 1 x 0,12 = \$120

Again, these \$120 will not be added. So,

I_3 = 1,000 x 1 x 0,12 = \$120

At the end of the third year, total interest, I, will be

120 + 120 + 120, i.e., 3 x 120 = \$360.

The total amount will then be \$1,360 (1,000 + 360).

This is usually called "Accumulated Value", "Final Value" or "Future Value" (C_n or FV).

So, under simple interest,

$C_n = C_0 + I$ (or FV = PV + I)

and

$I = C_0\, n\, i$ (or I = PV n i)

which means that

$C_n = C_0 + C_0\, n\, i$
$\mathbf{C_n = C_0\ (1+ni)}$

or

FV = PV + PV n i
FV = PV (1+ni).

So, under simple interest, having an interest rate of 12% per year or 6% per semester or 3% per quarter or 1% per month is the same: after (let's say) one year, the total amount of interest is the same, no matter if it is computed at 12% per year, 6% per semester, 3% per quarter or 1% per month. That is, under simple interest, the relationship between interest rates expressed in different periods is a *proportional relationship*: if the interest rate is, let's say, 12% per year, it will be

. 6% per semester (1/2 the period, ½ the interest rate)
. 3% per quarter (1/4 the period, ¼ the interest rate)
. 1% per month (1/12 the period, 1/12 the interest rate)
. and so on

2.2 - Compound interest. Interest rates under compound interest. Annual nominal interest rate (NOM) and annual effective rate (EFF).

Compound interest is characterized by the fact that interest produced in any period will earn interest in the forthcoming periods, i.e., there is interest on interest. Pay attention:

$I_1 = C_0 \times 1 \times i = C_0\, i$
$C_1 = C_0 + I_1 = C_0 + C_0 i = C_0\,(1+i)$

$I_2 = C_1 \times 1 \times i = C_1\, i$
$C_2 = C_1 + I_2 = C_1 + C_1 i = C_1\,(1+i) = C_0\,(1+i)^2$

$I_3 = C_2 \times 1 \times i = C_2\, i$
$C_3 = C_2 + I_3 = C_2 + C_2 i = C_2\,(1+i) = C_0\,(1+i)^3$

And so on. At the end of the n^{th} period we will have

$\mathbf{C_n = C_0\,(1+i)^n}$ (or $\mathbf{FV = PV\,(1+i)^n}$)

Note: from now on we will use more often PV and FV instead of C_0 and C_n.

<u>Example 5</u>

PV = \$1,000; n = 3 years; i = 12% (annual)

$I_1 = 1{,}000 \times 1 \times 0.12 = \$120$

After 1 year, the interest is \$120 which is then added to the initial amount of \$1,000. On the second year, the amount that will produce interest is \$1,120. So,

$I_2 = 1{,}120 \times 1 \times 0.12 = \$134.40$

Again, these \$134.40 are added to the \$1,120 that we had at the beginning of the second year, summing \$1,254.40. So,

$I_3 = 1{,}254.40 \times 1 \times 0.12 = \$150.53$

At the end of the third year the total amount will then be \$1,404.93 instead of \$1,360 under simple interest, as we saw on Example 4.

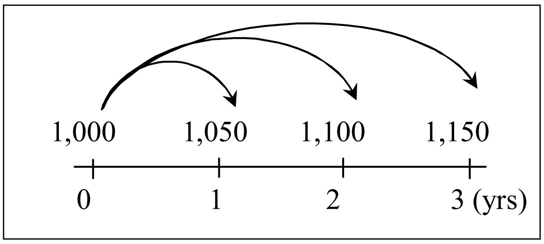
As we can see, now, under compound interest, the periodic interest (interest produced in any period) is not constant; instead, it grows – and it grows *geometrically*:

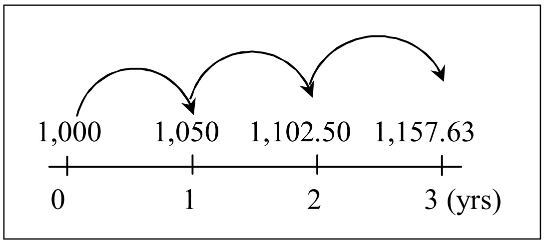
$I_1 = PV\ i$
$I_2 = I_1\ (1+i)$
$I_3 = I_2\ (1+i) = I_1\ (1+i)^2$
$I_4 = I_3\ (1+i) = I_1\ (1+i)^3$, and so on

It's easy to understand that, under compound interest, $\mathbf{FV = PV\ (1+i)^n}$ as we saw before.

Again, it is vital that *n* and *i* are expressed at the same period (i.e., if *i* is annual, *n* must be expressed in years; if it is interest rate for the month, *n* must be expressed in months; etc.).

Visually, simple interest (on the left) and compound interest (on the right) behave like the graphs below, both at a same 5% annual rate in a 3-year period. See the difference?





Now: what if the interest is computed more than once per year? In this case we need to know what really means "12% per year". In fact, now there is interest on interest. So,

1) Does "12% per year" mean that interest will be computed at, say, 1% per month (or 3% per quarter, 6% per semester)? If so, we must understand that at the end of one year we will have <u>more than 12%</u>, because there is interest on interest.

or

2) Does "12% per year" mean that, no matter how many times interest is computed during the year, at the end of the year we will earn 12%? If so, this means that the monthly interest rate (for instance) can't be 1%. It has to be <u>less than 1%</u>. If it was 1%, then at the end of one year we would have more than 12% because of the existence of interest on interest.

In the first situation, the 12% is called *annual nominal interest rate* (NOM) and we must understand that if we have two or more compoundings during one year, then the "effective" interest rate for one year is higher than 12%. And it will be higher and higher as we have more and more compoundings during the year.

In the second situation, the interest rate of 12% is called *annual effective interest rate*, EFF (no matter how many compoundings we have during the year, at the end of the year we have 12%). So, we can say that:

- **EFF** is the annual rate that already reflects the effect of compounding, no matter how many compoundings we have during the year;

- **NOM** is the annual rate as if there was no interest on interest (i.e., as if simple interest was used).

Let's assume from now on the following symbols:

> **i**: annual effective interest rate (EFF)
> $\mathbf{i_{(k)}}$: annual nominal interest rate, with k compoundings per year (NOM)
> $\mathbf{i_k}$: effective interest rate for the period 1/k of the year

So,

> **i = 12%** means that, no matter how many compoundings exist during the year, the interest corresponds to 12% at the end of the year.
>
> $\mathbf{i_{(12)}}$ **= 12%** means that the interest rate after one year would be 12% if there was no interest on interest during the year. However, there are 12 compoundings during the year (k=12), i.e., every month. This means that the monthly interest rate is 1%. So, the effective interest rate at the end of the year will be higher than 12% because there are 12 compoundings during the year.
>
> $\mathbf{i_{12}}$ **= 1%** means that the interest rate for the month is 1%. There is interest every month. So, at the end of the year, effective interest rate will be higher than 12%.

We must understand that under compound interest when we have a nominal interest rate it is absolutely crucial that we know the frequency of compoundings (i.e., how many compoundings we have per year) because that determines the annual effective interest rate.

Example 6

Annual nominal interest rate: 12% compounded every semester.

i) Effective interest rate for the semester?
ii) Effective interest rate for the month?
iii) Effective interest rate for the quarter?
iv) Annual effective interest rate?

We have $i_{(2)}$ = 12%. A vital information here is that **compoundings** happen **every semester**.

Why **vital**? Because

a) It's compound interest
b) 12% is annual nominal

In fact,

a) If it was simple interest, it wouldn´t matter the compounding frequency because there wouldn't exist interest on interest.

b) If the interest rate was annual effective frequency of compoundings would be no worry at all; and the interest rate being nominal, if compoundings happened with any other frequency, the period of compoundings would be vital.

So, in such a situation (compound interest and nominal interest rate) the most important interest rate is the interest rate for the same period of compoundings, computed using a *proportional relationship*. Any other interest rate, for any other period, must be computed from this one.

Back to our example: first of all, we must compute the interest rate for the semester, i_2. How? Using a *proportional relationship*. Why? Because 12% is nominal. This means that, if there was no interest on interest, the interest rate after one year would be 12%, i.e., the interest rate for the semester is 6%: $i_2 = 0{,}12 / 2 = 0{,}06$. This is the effective interest rate for the semester (question i)).

Now, to compute the interest rate for any other period, we must keep in mind that they must be computed in such a way that the interest rate for the semester is 6%. This means that, say, the monthly interest rate cannot be 1%, because 1% every month under compounding interest would lead to more than 6% at the end of the semester. So, the interest rate for the month has to be lower than 1%. How much, exactly? Well, it has to be i_{12} so that

$$\underbrace{(1+0{,}06)^1}_{|1 \text{ semester}|} = \underbrace{(1+ i_{12})^6}_{|6 \text{ months}|} \quad \textit{Equivalence relationship between interest rates}$$

Let's *read* this: we are looking for a monthly interest rate, i_{12}, so that the result is the same either if there is only 1 compounding at 6% after one semester or 6 monthly compoundings at i_{12}.

Notice that the "1" to which (1+0,06) is raised to means "1 semester" (because 6% is the interest rate for the semester); the "6" to which $(1+ i_{12})$ is raised to means "6 months" (because i_{12} is the interest rate for the month). The time period in both sides of the equation has to be the same (in this example, 1 semester = 6 months).

So, let's answer the questions:

i) $i_2 = 0{,}12/2 = 0{,}06$

ii) $(1+0{,}06)^1 = (1+ i_{12})^6 \Leftrightarrow i_{12} = 0{,}009759$ (in the first equation: 1 semester = 6 months)

iii) $(1+0{,}06)^1 = (1+ i_4)^2 \Leftrightarrow i_4 = 0{,}02956$ (in the first equation: 1 semester = 2 quarters)

iv) $(1+0{,}06)^2 = (1+ i)^1 \Leftrightarrow i = 0{,}1236$ (in the first equation: 2 semesters = 1 year)

2.3 - Day count basis (conventions to compute number of days between dates).

When computing interest on a day-to-day basis, several conventions can be used when matching the time unit of the values. Some of them are the following:

- ACT/365
- ACT/360
- 30/360

. "ACT" means that days are counted on a "real" calendar basis (ACTual days between both dates, starting date and ending date, taking into account if it is a leap year or a common year, i.e., 29 or 28 days in February, respectively;
. "30" means that it is assumed that every month has 30 days;
. "365" means that it is assumed that one year has 365 days;
. "360" means that it is assumed that one year has 360 days.

Besides, usually the first day of the period is not counted and the last one is.

Example 7

Deposit of €1.000 from 20th Jan 2016 to 14th Sep 2016; i = 8%. How much is the interest, assuming simple interest and

a) ACT/365
b) ACT/360
c) 30/360

(Note that 2016 was a leap year)

Counting the days:

	ACT	30
Jan	11	10
Feb	29	30
Mar	31	30
Apr	30	30
May	31	30
Jun	30	30
Jul	31	30
Aug	31	30
Sep	14	14
Total:	**238**	**234**

So,

a) ACT/365
I = 1.000 x 238/365 x 0,08 = €52,16

b) ACT/360
I = 1.000 x 238/360 x 0,08 = €52,89

c) 30/360
I = 1,000 x 234/360 x 0,08 = €52,00

EXTRA: Tax on interest

In Portugal there is a tax on interest. It used to be 20% but in 2010 the Government changed it to 21,5%, later to 25%, 26,5% and more recently 28%. The bank, itself, retains the tax and delivers it to the Treasury in behalf of the investor.

So, it is important to define precisely the interest rate in order to be clear if it is a "pre-tax" interest rate or an "after-tax" (or *net*) interest rate. Besides, we must pay attention that, every time interest is computed, tax is retained by the bank (not paid to investor). This means that, if compound interest is being used, the amount of interest that will be added for the next period will be only the "net interest", i.e., after-tax (72% of pre-tax interest rate).

Example 8

Suppose you want to invest €1.000 for 1 year. Which bank would you choose?

- Bank A: Annual nominal pre-tax rate: 3,765%; monthly compoundings
- Bank B: Annual effective net rate: 2,9%; monthly compoundings

(Tax rate on interest: 28%).

Bank A: we must pay attention that the announced interest rate is pre-tax and that this bank compounds interest every month. So, we must

1. Compute the interest rate for the month (proportional, because the announced interest rate is nominal). This interest rate is still pre-tax. So, we must then
2. Compute the net interest rate (for the month)

Thus,

1. Interest rate for the month (still pre-tax)
   $i_{12 \text{ pre-tax}} = 0{,}03765/12 = 0{,}003138$

2. Net monthly interest rate
   $i_{12 \text{ net}} = 0{,}72 \text{ x } 0{,}003138 = 0{,}002259$

So, if we choose Bank A, we will have one year later

$$FV = 1.000\ (1+0{,}002259)^{12} = €1.027{,}452 \approx €1.027{,}45$$

Bank B: this interest rate is already effective and net. Being net (i.e., after-tax), we don't have to worry about taxes; being effective, we don't have to worry about compoundings. So, we can compute FV directly:

$$FV = 1.000\ (1+0{,}029)^{1} = €1.029$$

Conclusion: Bank B is a little bit better.

Exercises

Simple interest

2.1 - Aleksandra made a deposit of $3,000 under these conditions: simple interest, interest rate of 5%, annual. How much interest will she receive 2 years later?

(I =$300)

2.2 - Brigita made a deposit of $1,000 at 6% per year (simple interest). How much will be the interest she will receive after 4 months?

(I=$20)

2.3 - Julius made a simple interest deposit of $5,000 at 4% annual. How much will be the interest he will receive after 150 days, assuming

i) Civil year (365 days)
ii) Commercial year (360 days)

(i: I=$82.19; ii: I=$83.33)

2.4 - Magdalena made a deposit of €10.000, at 5% annual interest rate, between 20th November 2014 and 20th March 2015 (simple interest). How much was the interest if the convention used was

i) ACT/365
ii) ACT/360
iii) 30/360

(i: I=€164,38; ii: I=€166,67; I=€166,67)

Compound interest

2.5 - Rugile made a deposit of €7.500 at 12% annual nominal interest rate (compound interest). How much was future value after 1 year, assuming

i) Monthly compounding
ii) Quarterly compounding
iii) Annual compounding

(i: FV=€8.451,19; ii: FV=€8.441,32; iii: FV=€8.400)

2.6 - Agnieszka made a deposit of $7,000 (compound interest). After 8 quarters, future value was $7,715.20. Compute

i) Annual nominal interest rate
ii) Annual effective interest rate

(i: $i_{(4)}$=4.8938%; ii) i=4.9843%)

2.7 - Magda made a deposit of $40,000 at 5% annual effective interest rate (compound interest) from which she received the future value of $48,620.25. For how long did she keep this deposit?

(n=4 years)

Interest rates

2.8 - Annual effective interest rate: 10%. Effective interest rates for

i) Semester
ii) Quarter
ii) Month

(i: i_2=4,8809%; ii: i_4=2,4114%; iii: i_{12}=0,7974%)

2.9 - Annual nominal interest rate: 10%; monthly compounding. Annual effective interest rate?

(i=10,4713%)

2.10 - Effective interest rate for the quarter: 3%

i) Annual nominal interest rate?
ii) Annual effective interest rate?

(i: $i_{(4)}$=12%; i=12,5509%)

2.11 - Annual nominal interest rate: 9%; quarterly compounding.

i) Effective interest rate for the month?
ii) Effective interest rate for the semester?
iii) Effective interest rate for the quarter?
iv) Annual effective interest rate?

(i: i_{12}=0,7444%; i_2=4,5506%; i_4=2,25%; i=9,3083%)

2.12 - Annual effective interest rate: 9%; quarterly compounding.

i) Effective interest rate for the month?
ii) Effective interest rate for the semester?
iii) Effective interest rate for the quarter?
iv) Effective annual interest rate?

(i: i_{12}=0,7207%; i_2=4,4031%; i_4=2,1778%; i=9%)

2.13 - Suppose you want to invest €10.000 for 1 year. Which bank would you choose? (Consider tax rate on interest is 25%).

- Bank A: Annual nominal net rate: 4,6%; quarterly compounding
- Bank B: Annual nominal pre-tax rate: 6%; monthly compounding
- Bank C: Annual effective net rate: 4,5%; monthly compounding
- Bank D: Annual effective pre-tax rate: 6,1%; monthly compounding

(Bank A: €10.468; Bank B: €10.459,40; Bank C: €10.450; Bank D: €10.454,38)*

2.14 - Interest rate: 6% annual nominal compounded every semester. Compute the following interest rates:

i) Effective for the month
ii) Annual effective

(i: i_{12} = 0,4939%; ii: i = 6,09%)

2.15 - Interest rate: 3%, effective for the quarter. Compute the following interest rates:

i) Effective for the semester
ii) Effective for the month
iii) Annual nominal, compounded every quarter
iv) Annual effective

(i: i_2 = 6,09%; ii: i_{12} = ,9902%; iii: $i_{(4)}$ = 12%; iv: i = 12,5509%)

2.16 - Interest rate: 7% annual effective. Compute the following interest rates:

i) Annual nominal compounded every quarter
ii) Annual nominal compounded every month
iii) Annual nominal compounded every year

(i: $i_{(4)}$ = 6,8234%; ii: $i_{(12)}$ = 6,785%; iii: i = 7%)

2.17 - Interest rate: 12% annual nominal, pre-tax, compounded every month. Compute the annual effective net (after tax) interest rate. (Tax rate: 28%)

(i_{net} = 8,9905%)

2.18 - Interest rate: 12% annual effective, pre-tax. Interest is computed every month. Compute the annual effective net (after tax) interest rate. (Tax rate: 28%)

(i_{net} = 8,5135%)

3. EQUIVALENCE BETWEEN CASH-FLOWS

3.1 - Equivalence: compounding and discounting under simple interest and compound interest. Weaknesses of discounting under simple interest and strengths of discounting under compound interest.

According to the Golden Rule of Financial Calculus (page 5), in order to correctly compare or operate with cash-flows we must express all of them at one same moment. We already know how to express any amount, *Present Value* (PV), at a later moment (remember that we call that future amount *Future Value*, FV). As we saw, under simple interest, FV = PV (1+ni); under compound interest, FV = PV $(1+i)^n$. But there are many situations that require us to express a certain amount at an earlier moment, instead of a later moment. Why should we do this? And how can we do it?

We may need to do this if, for instance, a future debt is to be paid right now (or at any earlier moment than it should be). And how can we do it? Well, first of all, it is understandable that if the debt is going to be paid earlier than it was supposed to, then the amount to be paid will be lower. In fact, it is fair that the interest to pay will be lower. The question is: how to compute the new amount to pay? We can conceive at least two ways to do this: under simple interest or under compound interest[1].

This (expressing a cash-flow at an earlier moment) is usually called "discount" (while expressing a cash-flow at a later moment is "compound").

It's exactly by *compounding* and *discounting* that **equivalence** between two or more cash-flows, expressed at different moments, can be established. Remember that we need to do this because of time value of money (we need to express every cash-flow at one same moment).

Example 9

Debt to be paid 2 years from now, €1.000 (interest included); interest rate: 6% (annual effective). How much should the debtor pay today if

a) Simple interest is used?
b) Compound interest is used?

Let's see:

a) Under simple interest what we usually do is compute the interest like this:
I = FV.n. i ; as PV = FV – I, then PV = FV – FV.n.i ⇔ **PV = FV (1-ni)** [2]

In this case,

I = 1.000 x 2 x 0,06 = €120. So, the debtor should pay PV = 1.000 – 120, or
PV = 1.000 (1 - 2 x 0,06)⇔ PV = €880.

But we should notice that this approach embodies a serious mistake. The interest should not be computed upon 1.000. In fact, what happens is that those 1.000 euro *include* the interest of another (shorter) amount, for 2 years, at 6%, which is a huge difference. However, this is an approach that is often used in real life[3].

[1] Actually, there are two ways to do it under simple interest plus two ways to do it under compound interest, but we will study only one of each.

[2] This is called *Bank (Simple) Discount*. Better than this is the so called *True (Simple) Discount*, where I = PV.n.i. This approach is much more accurate. This way, we have PV = FV - I ⇔ PV = FV – PV.n.i ⇔ PV = FV / (1+n.i).

[3] At least, in Portugal. we have a financial instrument called *letra de câmbio* as an everyday example of this situation, with some other specific aspects (similar to *wechsel* in Germany, *weksel* in Poland, *vekselis* in Lithuania, *epitagi* in Greece and *kambiyo senetleri* in Turkey – well, hopefully…).

Now, imagine that the creditor deposits today the €880 on a bank also under simple interest, at the same interest rate and for the same period of time. We easily understand that he or she will not get €1.000 one year later. In fact,
880 (1 + 2 x 0,06) = €985,60 (< €1.000).

b) Under compound interest we would simply do this: **PV = FV $(1+i)^{-n}$**

   In this case, PV = 1.000 $(1+0,06)^{-2}$ = €890

This approach (compound interest) is much more *accurate.* Indeed, notice that now, if the creditor puts this money on a bank, also under compound interest, at the same interest rate and for the same period of time, he or she will get exactly €1.000 one year later (pure mathematics…):

   890 $(1+0,06)^2$ = €1.000

Previous example shows how weak is discounting under simple interest and how strong is discounting under compound interest.

In fact, under simple interest, because of the way interest is computed (on FV, indeed), it could happen (in theory) that PV = 0, which is nonsense. This would happen if ni=1. Worse: it could even happen (in theory) that PV < 0. This would happen if ni>1. This never happens under compound interest.

So, discounting using simple interest (Bank Simple Discount) is only acceptable for short period and low interest rate operations. Discounting using compound interest, on the contrary, is immune to these variables. It always leads to perfect equivalence, no matter if for short or long period and low or high interest rates. Check this example:

Example 10

Debt to pay 5 years from now, \$1,000; interest rate: 20%, annual. How much should de debtor pay today if

a) Simple interest is used?
b) Compound interest is used?

Let's see:

a) Simple interest: PV = FV (1 - ni) = 1.000 (1 - 5 x 0.20) = \$0
   This is nonsense… It would be something like this: "OK, I owe you \$1,000 that I must pay 5 years from now. I would like to pay this debt right now. The interest rate is 20% per year and we will use simple interest. Let´s compute how much must I pay you now, in order to be equivalent. Oh!... It's zero! So, my debt is paid, actually! Bye-bye!"
b) Compound interest: PV = FV $(1 + i)^{-n}$ = 1,000 $(1 + 0.20)^{-5}$ = \$401.88
   See how this makes sense: in this situation, the creditor will receive now \$401.88. This will be a perfect equivalence if he or she invests this amount for 5 years at 20% per year and gets \$1,000. Let's check:
   401.88 $(1 + 0,20)^5$ = \$1,000. Perfect!

   Now: imagine that instead of n = 5 years it was n = 6 years and/or instead of i = 20% it was i = 25% (i.e., that it was a longer period of time and/or higher interest rate operation). Can you see what would happen?

Previous examples show that discounting under simple interest is not really a good equivalence solution, while discounting under compound interest is perfect. Indeed, simple discount is always bad but as we saw on Example 10, it can even be not acceptable at all under certain circumstances (long-term and/or high interest rate operations). It is a very weak equivalence solution. It is only acceptable for short-term operations and with low interest rates. On the contrary, compound discount is a perfect equivalence solution. It always leads to a perfect equivalence, both for short-term or long-term periods and for low or high interest rates.

3.2 - Equivalence factors.

As we just saw, when we deal with money and, for some reason, we need to compare or operate with cash-flows expressed at different moments, we must keep in mind that the first thing we have to do is express all those cash-flows at one same moment (that moment is called *focal date*).

This may require one or more cash-flows to be compounded, other or others to be discounted. Eventually one or more will need nothing to be done. This will be the situation if that or those cash-flows are already expressed at the focal date.

Compounding and discounting can be done using simple interest or compound interest. Compound interest is much more accurate and strong, so it is much more used in real life situations.

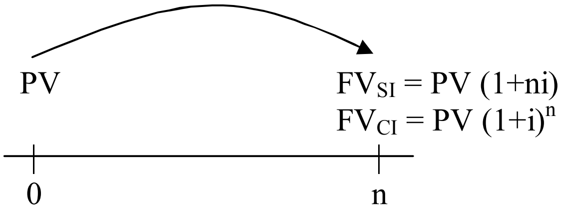
Until this moment, we only studied how to compound and discount **one single** cash-flow at a time. Starting next chapter we will study how to do it for a set of cash-flows. As we will see, if that set of cash flows fulfills two conditions, computing Future Value and Present Value of all of them at once (no matter how many they are) will be very easy.

For now, let's just summarize how to compound and discount one single cash-flow. As we saw,

	Future Value	Present Value
Simple Interest (SI)	$FV_{SI} = PV\ (1+ni)$	$PV_{SI} = FV\ (1–ni)$
Compound Interest (CI)	$FV_{CI} = PV\ (1+i)^{n}$	$PV_{CI} = FV\ (1+i)^{-n}$

Using a diagram view,

Compound



Discount

$PV_{SI} = FV\ (1–ni)$ FV
$PV_{CI} = PV\ (1+i)^{-n}$

0 n

Notice that to compute FV we just multiply PV by **(1+ni)** or by **$(1+i)^n$**; to compute PV, we just multiply FV by **(1–ni)** or by **$(1+i)^{-n}$**. These are what we can call *factors of equivalence*. In fact, we only need to *multiply* PV or FV by the correct *factor* to get the *equivalent* cash-flow at a later moment (FV) or at an earlier moment (PV), respectively.

Please note that these factors are applied to *one single cash-flow*. If we have several cash-flows, we must apply the correct factor to every cash-flow, *one by one*.

In summary, we have these *factors of equivalence*[4]:

	Compounding factor (CF)	Discounting factor (DF)
Simple Interest (SI)	$CF_{SI} = (1+ni)$	$DF_{SI} = (1–ni)$
Compound Interest (CI)	$CF_{CI} = (1+i)^{n}$	$DF_{CI} = (1+i)^{-n}$

Finally, let's get our ideas straight: to establish equivalence between cash-flows we need to follow these three steps:

1. Understand between which cash-flow(s), on one hand, and which other cash-flow(s), on the other hand, we want to establish equivalence *(question: **what?**)*.
2. Define the moment when the equivalence is to be established, i.e., the moment to when each and every cash-flow will be translated *(focal date*) *(question: **when?**)*.
3. Define how the equivalence will be established, i.e., if assuming simple interest or compound interest *(question: **how?**)*.

Example 11

John must pay €5.000 on 20th January 2017 plus €10.000 on 20th July 2017. These amounts include interest at 10% (annual nominal, quarterly compounded). He intends to replace these payments by other two, as follows: €X to be paid on 20th July 2016 and €2X on 20th April 2017. How much must he pay in these days assuming the same interest rate, the convention 30/360 to compute days between dates and

a) Simple interest
b) Compound interest

Let's start by representing the situation in a diagram:

X	5.000	2X	10.000
20th Jul 2016	20th Oct 2016	20th Jan 2017	20th Apr 2017	20th Jul 2017

[4] Actually there are other equivalence factors but they are not so much used, at least in Portugal.

We want to replace those two initial payments (€5.000 on the 20th January 2017 plus €10.000 on the 20th July 2017) by another two payments (€X on the 20th July 2016 plus €2X on the 20th April 2017). In other words, we want to *establish equivalence* between [5.000 + 10.000], on one hand, and [X + 2X], on the other hand.

How do we do it? Well, answering those three questions we just talked about: *what*, *when*, *how*.

1) **What?**

We want to replace [5.000+10.000] by [X+2X] i.e., establish equivalence between [5.000+10.000] and [X+2X].

So, we can start by writing this equation, still unfinished:

$$\underbrace{5.000}_{20th\ Jan\ 2017} + \underbrace{10.000}_{20th\ Jul\ 2017} = \underbrace{X}_{20th\ Jul\ 2016} + \underbrace{2X}_{20th\ Apr\ 2017}$$

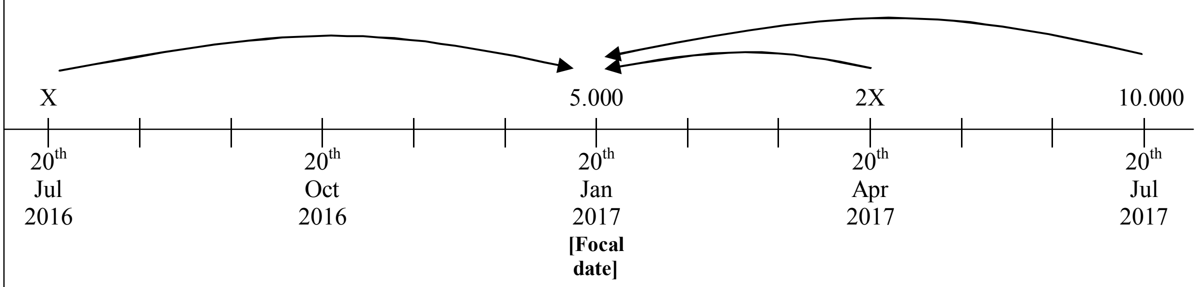
Why is this equation still unfinished? Because, as it is, not all the cash-flows are expressed at one same moment. So, we need to define this moment *(focal date)*. Here comes the second question: **when?**

2) **When?**

We need to define the focal date, i.e., the date when we want every cash-flow to be expressed. Let it be, for instance, 20th January 2017[5]. This way, we need to

- Compound X for 2 quarters (or 6 months)
- Do nothing with 5.000
- Discount 2X for 1 quarter (or 3 months)
- Discount 10.000 for 2 quarters (or 6 months)

In a diagram view,



Once defined the focal date, we need to compound or discount the cash-flows. As we know, we can do it in one of two ways. Here comes the third question.

3) **How?**

In question a), using simple interest; in question b), using compound interest. So, we can now finish the equation above just using the correct factors, as follows:

5 An important topic here is that the focal date is irrelevant in compound interest and is absolutely vital in simple interest. In other words, one or another focal date doesn't affect equivalence in compound interest (which is great!) but every time focal date changes in simple interest, equivalence changes as well (which is really bad). This is another weakness of simple interest and another strength of compound interest.

a) Simple interest:

$$\underbrace{\underbrace{5.000}_{20th\ Jan\ 2017} + \underbrace{10.000}_{20th\ Jul\ 2017}\left(1-\frac{6}{12}x0,10\right)}_{20th\ Jan\ 2017} = \underbrace{\underbrace{X}_{20th\ Jul\ 2016}\left(1+\frac{6}{12}x0,10\right)}_{20th\ Jan\ 2017} + \underbrace{\underbrace{2X}_{20th\ Apr\ 2017}\left(1-\frac{3}{12}x0,10\right)}_{20th\ Jan\ 2017}$$

$$5.000+9.500=1,05X+1,95X$$

$$14.500=3X$$

$$X=4.833,33$$

[As convention 30/360 is used, we can consider that every month has 30 days. In other words, we can use months, because every date is the 20^{th} of some month].

So, if simple interest is used, X = €4.833,33 (to be paid on 20^{th} July 2016) and 2X = €9.666,67 (to be paid on 20^{th} April 2017).

b) Compound interest:

Now it is vital to keep in mind that the interest rate, 10%, is annual nominal quarterly compounded. So, we need to compute the interest for the quarter. It is $i_4 = 0,10/4 = 0,025$. So,

$$\underbrace{\underbrace{5.000}_{20th\ Jan\ 2017} + \underbrace{10.000}_{20th\ Jul\ 2017}\left(1+0,025\right)^{-2}}_{20th\ Jan\ 2017} = \underbrace{\underbrace{X}_{20th\ Jul\ 2016}\left(1+0,025\right)^{2}}_{20th\ Jan\ 2017} + \underbrace{\underbrace{2X}_{20th\ Apr\ 2017}\left(1+0,025\right)^{-1}}_{20th\ Jan\ 2017}$$

$$5.000+9.518,14=1,050625X+1,95122X$$

$$14\ 518,14=3,001845X$$

$$X=4.836,41$$

So, if compound interest is used, X = €4.836,41 (to be paid on 20^{th} July 2016) and 2X = €9.672,82 (to be paid on 20^{th} April 2017).

As you can imagine, in real life we may need to establish equivalence between many (even hundreds of) cash-flows. In such situations, it may be really painful to establish the equivalence if we need to do it individually, i.e., cash-flow by cash-flow, one at a time. However, if one or two conditions exist, establishing the equivalence becomes very, very simple. Luckily, in real life this happens very often, as we will see later, on Chapters 4 and 5.

Exercises

Equivalence between cash-flows

3.1 – A few years ago, Asuman took a loan at 10% annual interest rate (effective). Because of this loan, she must pay €10.000 euro 3 years from now. Asuman intends to pay her debt today. How much must she pay today if:

i) Simple interest is used?
ii) Compound interest is used?

(i: €7.000 if focal date is moment 0; €7.692,31, if focal date is moment 3; ii: €7.513,15, no matter the focal date)

3.2 - Ania must pay Karolina two debts, as follows:

. €3.000 euro, 6 months from now
. €5.000 euro, 10 months from now

Ania wants to pay both debts 2 months from now. Assuming 12% annual effective interest, how much must she pay if we consider:

i) Simple interest?
ii) Compound interest?

(i: €7.480 if focal date is moment 2; €7.469,39 if focal date is moment 0; €7.500 if focal date is moment 6; €7518,52 if focal date is moment 10; ii: €7.524,94, no matter the focal date)

3.3 - Ruta must pay Toma €25.000 euro 4 months from now. However, Ruta intends to replace this debt by two, as follows: one to be paid today and the other 9 months from now. The amount of the last one (to be paid 9 months from now) is twice the amount to be paid today. Both agree with this replacement, using the interest rate of 9%, annual nominal, monthly compounded. How much must Ruta pay today and 9 months from now,

i) If simple interest is used?
ii) If compound interest is used?

(i: €8.460,24 today, plus €16.920,48 nine months from now if focal date is moment 4; ii: €8.454,52 today, plus €16.909,04 nine months from now, no matter the focal date)

3.4 - Same as previous, but assume that the interest rate is 9%, annual nominal, quarterly compounded.

(i: €8.460,24 today, plus €16.920,48 nine months from now if focal date is moment 4; ii:€8.453,65 today, plus €16.907,30 nine months from now, no matter the focal date)

3.5 - A firm owes today the following amounts to the same supplier:

Amount	Date
€12.500	1 month from now
€3.500	7 months from now
€8.000	14 months from now

That firm intends to replace these three debts by two equal payments, the first one to be paid 5 months from now and the other one year from now. Assuming the interest rate is 6%, annual nominal monthly compounded, compute these two new payments in the following scenarios:

i) Scenario 1: Simple interest; focal date: 6 months from now
ii) Scenario 2: Simple interest; focal date: today
iii) Scenario 3: Compound interest; focal date: 6 months from now
iv) Scenario 4: Compound interest; focal date: today

(i: €12.139,24 ; ii: €12.143,60; iii: €12.141,27; iv: €12.141,27)

3.6 - Same as previous, but assume that the interest rate is 6%, annual nominal, quarterly compounded. *(i: €12.139,24 ; ii: €12.143,60; iii: €12.140,56; iv: €12.140,56)*

3.7 - Same as exercise 3.5, but assume that the interest rate is 6%, annual effective.

(i: €12.139,24 ; ii: €12.143,60; iii: €12.137,44; iv: €12.137,44)

4. ANNUITIES

4.1 - Definition of annuity. Important concepts. Types of annuities.

Note: *from now on we will assume that compound interest is used*

Annuity: sequence of payments (PMT), usually of the same amount, made at equal intervals of time (i.e., always with the same frequency).

Examples of annuities: payments of house rents, mortgages, insurance, interest payments on bonds, payments on credit purchases, etc..

There are some **important concepts** that we must pay attention to:

- **Payment interval** or **annuity period**: time lapse between any two consecutive payments. It can be any period, but it has to be constant, always the same – month, quarter, semester, year, whatever. But always the same.

- **Origin of the annuity**: moment located one period *(i.e., one payment interval)* before the moment when the first payment occurs.

- **Future value of an annuity**: value (sum) of all payments, referred to the moment when the last payment occurs. It is usually represented by $\mathbf{FV_A}$. Also called **Accumulated value** of an annuity.

- **Present value of an annuity**: value (sum) of all payments, referred to the origin of the annuity. It is usually represented by $\mathbf{PV_A}$. Also called **Discounted value** of an annuity.

- **Term of the annuity:** total duration of the annuity

We can consider many **types of annuities**, according to different criteria. For instance,

According to the payment amount:

- **Constant payments:** if all the payments of the annuity are the same amount (represented by **PMT**).
- **Variable payments:** if not all the payments of the annuity are the same amount.

According to the term of the annuity:

- **Annuity certain**: when the end of the annuity is pre-determined (known). Example: bond interest payments.
- **Contingent annuity**: when the end of the annuity depends on some uncertain event (is unknown). Example: insurance.

According to the moment when payments are made:

- **Ordinary annuity**: when each payment is made at the end of the corresponding payment interval.
- **Annuity due**: when each payment is made at the beginning of the corresponding payment interval.
- **Deferred annuity**: when there are some periods of delay until the first payment occurs.

According to the period of the interest rate and the period of the annuity:

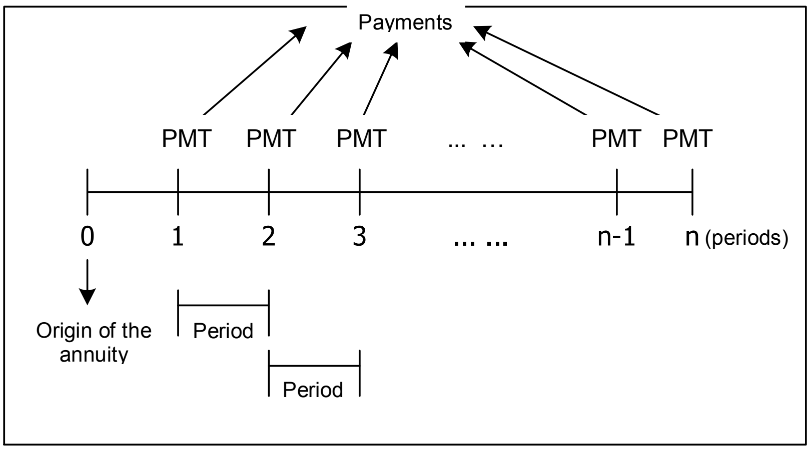
- **Simple annuity**: when the payment interval and the interest rate period are the same.
- **General annuity**: when the payment interval and the interest rate period are different.

We will use the following **notation** on annuities:

- **PMT** (or simply **p**): periodic payment (assumed constant)
- **n**: number of payments
- **i**: interest rate, referred to the same period of the payment interval
- $\mathbf{FV_A}$: future value of the annuity
- $\mathbf{PV_A}$: present value of the annuity

For now we will only consider annuities with all payments equal (the same amount) and whose period matches the period of the interest rate, i. e., simple and constant annuities. Later (section 4.5) we will discuss general annuities, still only with constant payments.

According to this, an annuity can be represented this way:

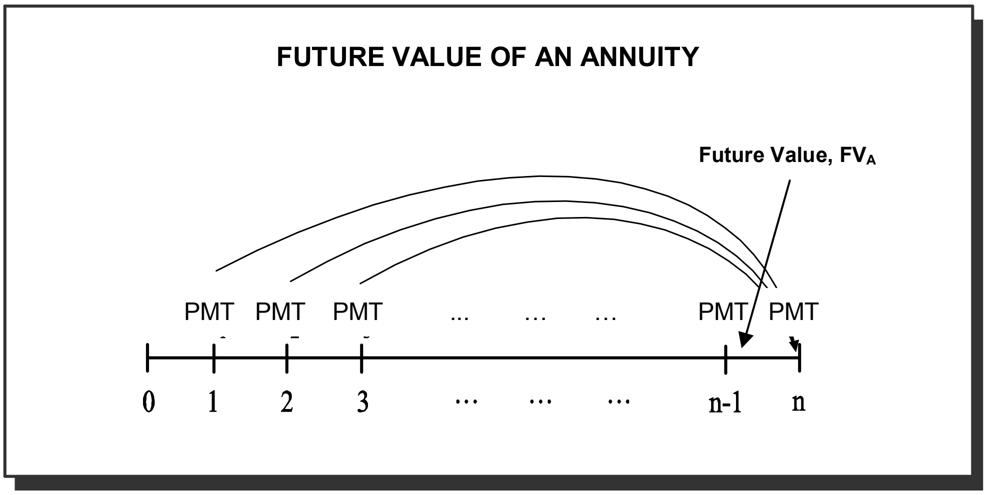


In this case, we have a *certain*, *ordinary* and *simple* annuity, assuming that we know the number of payments, *n*, the annuity starts at moment 0 and the interest rate period is the same as the annuity's.

Important remark: *this is just a symbolic representation of an annuity. It is not mandatory that the first payment occurs at moment 1 and the last payment at moment n. In other words, the origin of the annuity does not have to be always moment 0 and the last payment does not have to occur always at moment n.*

4.2 - Future value of a simple annuity.

As we saw, **future value of an annuity** is the value (sum) of all payments at the moment when the last payment occurs. This means that all of them must be compounded to that moment (well, not exactly *all* of them, actually: the last one is not compounded, because it occurs exactly at that moment). In a diagram,



Analytically, we have:

$$\underbrace{FV_A}_{Mom.\ n} = \underbrace{\underbrace{PMT}_{Mom.1}(1+i)^{n-1}}_{Mom.\ n} + \underbrace{\underbrace{PMT}_{Mom.\ 2}(1+i)^{n-2}}_{Mom.\ n} + \underbrace{\underbrace{PMT}_{Mom.\ 3}(1+i)^{n-3}}_{Mom.\ n} + \ldots + \underbrace{\underbrace{PMT}_{Mom.\ n\text{-}2}\ (1+i)^{2}}_{Mom.\ n} + \underbrace{\underbrace{PMT}_{Mom.\ n\text{-}1}\ (1+i)^{1}}_{Mom.\ n} + \underbrace{PMT}_{Mom.\ n}$$

After some mathematics, we get

$$\boxed{FV_A = PMT\frac{(1+i)^n - 1}{i}}$$

Usually the fraction above is represented by $s_{\overline{n|}\,i}$, i.e., $\boxed{s_{\overline{n|}\,i} = \frac{(1+i)^n - 1}{i}}$

So, we may write $\boxed{FV_A = PMT.s_{\overline{n|}\,i}}$

There are three important notes about this formula:

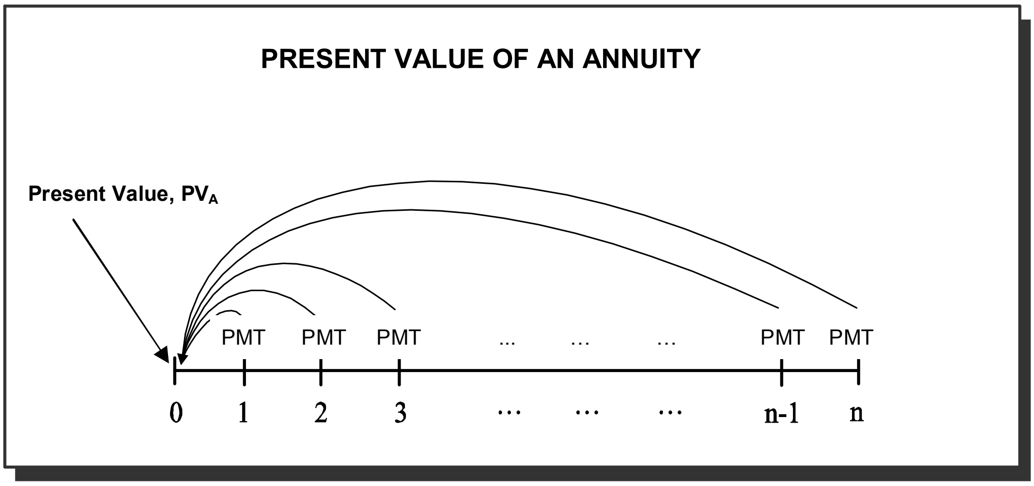
1. **FV_A:** is the value of all payments at one particular moment: the moment when the last payment occurs;
2. **n:** is the number of payments (*payments*, not *periods*!);
3. **i:** is the interest rate correctly converted to the same period of the annuity.

What may we need to compute about this? One of four things:

1. **FV_A:** if this is the unknown, no problem will arise; it is very easy to compute (analytically or using a financial calculator or a spreadsheet);
2. **PMT:** again, if this is the unknown, no problem; it is also easy to compute, either way;
3. **n:** if this is the unknown, we can solve the equation analytically (using logarithms) or using a financial calculator or a spreadsheet. But we must keep in mind that n represents the number of payments, so it must be an integer. This means that in some problems we may need to make some kind of adjustment to one payment. We will see this on Example 14;
4. **i:** if this is the unknown, we can't solve it analytically. We need a financial calculator or a spreadsheet. We will see this on Example 15.

4.3 - Present value of a simple annuity.

As we saw, **present value of an annuity** is the value (sum) of all payments at the origin of the annuity. This means that all of them are discounted to that moment (including the first one, because according to the concept of origin, it must be discounted one period). In a diagram,



Analytically, we have:

$$\underbrace{PV_A}_{Mom.\ 0} = \underbrace{\underbrace{PMT}_{Mom.1}(1+i)^{-1}}_{Mom.\ 0} + \underbrace{\underbrace{PMT}_{Mom.\ 2}(1+i)^{-2}}_{Mom.\ 0} + \underbrace{\underbrace{PMT}_{Mom.\ 3}(1+i)^{-3}}_{Mom.\ 0} + \ldots + \underbrace{\underbrace{PMT}_{Mom.\ n-2}(1+i)^{-(n-2)}}_{Mom.\ 0} + \underbrace{\underbrace{PMT}_{Mom.\ n-1}(1+i)^{-(n-1)}}_{Mom.\ 0} + \underbrace{\underbrace{PMT}_{Mom.\ n}(1+i)^{-n}}_{Mom.\ 0}$$

After some mathematics, we get

$$\boxed{PV_A = PMT\frac{1-(1+i)^{-n}}{i}}$$

Usually the fraction above is represented by $a_{\overline{n|}\,i}$, ie, $\boxed{a_{\overline{n|}\,i} = \frac{1-(1+i)^{-n}}{i}}$

So, we may write $\boxed{PV_A = PMT.a_{\overline{n|}\,i}}$

Again, there are three important notes about this formula:

1. **PV_A:** is the value of all payments at one particular moment: the *origin* of the annuity;
2. **n:** is the number of payments (*payments*, not *periods*!);
3. **i:** is the interest rate correctly converted to the same period of the annuity

What may we need to compute about this? One of four things:

1. **PV_A:** if this is the unknown, no problem will arise; it is very easy to compute (analytically or using a financial calculator or a spreadsheet);
2. **PMT:** again, if this is the unknown, no problem; it is also easy to compute, either way;
3. **n:** if this is the unknown, we can solve the equation analytically (using logarithms) or using a financial calculator or a spreadsheet. But we must keep in mind that n represents the number of payments, so it must be an integer. This means that in some problems we may need to make some kind of adjustment to one payment (Example 14);
4. **i:** if this is the unknown, we can't solve it analytically. We need a financial calculator or a spreadsheet. We will see this on Example 15.

4.4 - Value of an annuity at any moment.

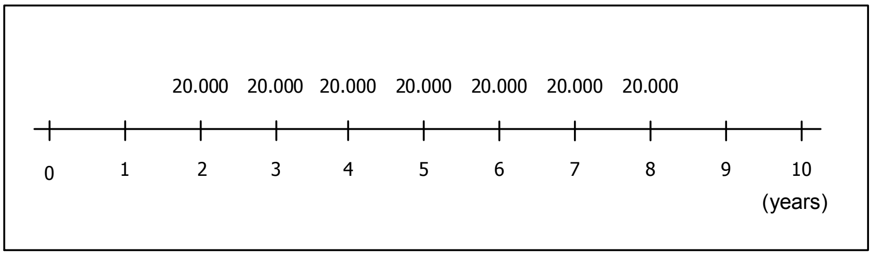
We can easily compute the value of an annuity at any moment. In fact, we must remember two things:

1. What we have just seen, FV_A and PV_A, are the single amounts equivalent to the *n* payments of the annuity in two particular moments: the moment when the last payment occurs (FV_A) and the moment we called *origin* of the annuity (PV_A);

2. We are using compound interest; so, we may establish the equivalence at any moment we want because equivalence is perfect, as we saw on Chapters 2 and 3. If we correctly compound or discount FV_A or PV_A, we will get another equivalent amount at another moment. But keep in mind that FV_A and PV_A are single amounts; so, we just multiply by $(1+i)^n$ or $(1+i)^{-n}$ to compound or discount them.

Example 12

Find the value of the following annuity at the indicated moments, assuming the annual effective interest rate of 15% (values in €):



a) Value at moment 8
b) Value at moment 10
c) Value at moment 1
d) Value at moment 0
e) Value at moment 2
f) Value at moment 9
g) Value at moment 6

Explanation

First, notice that this is a simple annuity (payment interval is the same as interest rate period) with 7 payments.

a) Moment 8 is a particular moment: it is the moment when the last payment occurs. So, it's very easy to compute the value at that moment: it is exactly FV_A.

$$\mathbf{V_8 = FV_A = 20.000\ s_{\overline{7|}0,15} = €221.335,98}$$

What could this be, in real life?
Something like this, for instance: you intend to save some money in the next years. If you save €20.000 every year, starting 2 years from now (assuming that today is moment 0) and for 7 years (i.e., you will make your last deposit 8 years from now) and the interest rate is 15% annual effective, how much will you have in your bank account at that exact moment, i.e., 8 years from now, immediately after you make the 7^{th} (last) deposit)? Answer: €221.335,98.

b) To compute the value of the annuity at moment 10 we can simply compound V_8 (FV_A) two years, from moment 8 to moment 10:

$$\mathbf{V_{10} = V_8\,(1+i)^2 = FV_A\,(1+i)^2 = 20.000\ s_{\overline{7|}0,15}\,(1+0,15)^2}$$

$$\mathbf{V_{10} = 221.335,98\,(1+0,15)^2 = €292.716,84}$$

Think: what could this be, in real life, assuming the example given on question a)?

c) Moment 1 is, we can say, a particular moment: it's the origin of this annuity; so, we simply want PV_A:

$$\mathbf{V_1 = PV_A = 20.000\ a_{\overline{7|}0,15} = €83.208,39}$$

Now: what could this be, in real life?
Something like this, for instance: today (moment 0) you have 7 debts to pay, €20.000, every year, starting 2 years from now. How much should you pay instead, if you decide to pay all of them at once, 1 year from now? Answer: €83.208,39.

d) To compute the value of the annuity at moment 0 we can simply discount V_1 one year, from moment 1 to moment 0:

$$\mathbf{V_0 = V_1\,(1+i)^{-1} = PV_A\,(1+i)^{-1} = 20.000\;a_{\overline{7|}0,15}\,(1+0,15)^{-1}}$$

$$\mathbf{V_0 = 83.208{,}39\,(1+0,15)^{-1} = €72.355{,}13}$$

Think: <u>what could this be, in real life</u>, assuming the example given on question c)?

We could also, for instance, discount V_8 eight years, from moment 8 to moment 0:

$$\mathbf{V_0 = V_8\,(1+0,15)^{-8} = 221.335{,}98\,(1+0,15)^{-8} = €72.355{,}13}$$

e) To compute the value of the annuity at moment 2 we can simply compound V_1 one year:

$$\mathbf{V_2 = V_1\,(1+i)^{1} = PV_A\,(1+0,15)^{1} = 20.000\;a_{\overline{7|}0,15}\,(1+0,15)^{1}}$$

$$\mathbf{V_2 = 83.208{,}39\,(1+0,15)^{1} = €95.689{,}65}$$

We could compute this value in other ways. For instance, we could discount V_8 six years:

$$\mathbf{V_2 = V_8\,(1+i)^{-6} = FV_A\,(1+i)^{-6} = 221.335{,}98\,(1+0,15)^{-6}}$$
$$\mathbf{V_2 = €95.689{,}65}$$

f) To compute the value of the annuity at moment 9 we can compound V_8 one year:

$$\mathbf{V_9 = V_8\,(1+i)^{1} = 221.335{,}98\,(1+0,15)^{1} = €254.536{,}38}$$

We could compute this value in other ways. For instance, we could discount V_{10} one year:

$$\mathbf{V_9 = V_{10}\,(1+i)^{-1} = 292.716{,}84\,(1+0,15)^{-1} = €254.536{,}38}$$

g) We can get the value of the annuity at moment 6 in several ways. For instance:

$$\underbrace{\mathbf{V_6}}_{\textit{mom. 6}} = \underbrace{\mathbf{20.000s_{\overline{5|}\,0,15}}}_{\substack{\textit{value of the first 5}\\ \textit{payments, expressed}\\ \textit{at mom. 6 (moment}\\ \textit{when the last of them}\\ \textit{occurs), i.e., FV of}\\ \textit{this annuity)}}} + \underbrace{\mathbf{20.000a_{\overline{2|}\,0,15}}}_{\substack{\textit{value of the last 2}\\ \textit{payments, expressed}\\ \textit{at mom. 6 (the origin}\\ \textit{of this annuity), i.e.,}\\ \textit{PV of this annuity}}}$$

$$\mathbf{V_6 = 134.847{,}63 + 32.514{,}18 = €167.361{,}81}$$

Another way to compute V_6:

$$\underbrace{\mathbf{V_6}}_{\text{mom. 6}} = \underbrace{\underbrace{\mathbf{20.000s_{\overline{4|}0,15}}}_{\substack{\text{value of the first 4}\\ \text{payments, at mom. 5}\\ \text{(moment when the}\\ \text{last one of them occurs)}}} \mathbf{(1+0,15)}}_{\substack{\text{value of the first 4 payments}\\ \text{at moment 6}}} + \underbrace{\mathbf{20.000}}_{\substack{\text{value of th 5}^{th}\\ \text{payment, which}\\ \text{occurs at mom. 6}\\ \text{(so, we don't need}\\ \text{to compound or}\\ \text{discount it)}}} + \underbrace{\mathbf{20.000a_{\overline{2|}0,15}}}_{\substack{\text{value of the last 2}\\ \text{payments, expressed}\\ \text{at moment 6 (origin}\\ \text{of this annuity)}}}$$

$$\mathbf{V_6 = 114.847,63 + 20.000 + 32.514,18 = €167.361,80}$$

We could also say that $V_6 = V_{10}\,(1+0,15)^{-4}$:

$$\mathbf{V_6 = 292.716,84\ (1+0,15)^{-4} = €167.361,80}$$

or that $V_6 = V_1\,(1+0,15)^{5}$:

$$\mathbf{V_6 = 83.208,39\ (1+0,15)^{5} = €167.361,79}$$

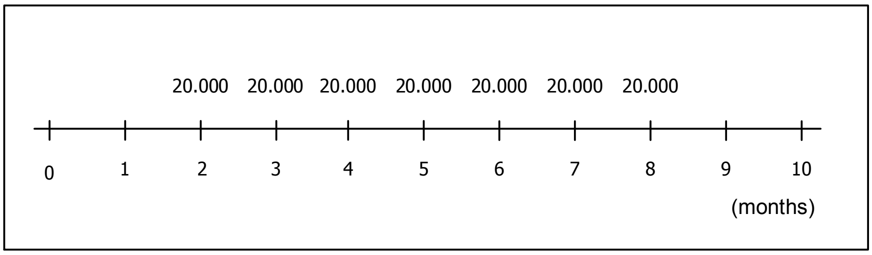
4.5 - Extension for general annuities.

As we saw, a **general annuity** means that the payment interval (period of the annuity) and the interest rate period are not the same. We solve this very easily: we just start by computing the interest rate for the same period of the annuity, using proportional or/and equivalence relationships, as mentioned in Chapter 2. Once done, we will have a simple annuity. From this moment on, we solve problems just like before (the rationale is the same).

<u>Example 13</u>

Find the value of the following annuity at the indicated moments, assuming that the interest rate is 12%

i) Annual effective
ii) Annual nominal, monthly compounded



a) Value at moment 8
b) Value at moment 10
c) Value at moment 1
d) Value at moment 0
e) Value at moment 2

<u>Explanation</u>

First, notice that this is a general annuity (payment interval: month; interest rate: annual). It has 7 payments, starting at moment 2 and ending at moment 8.

The first thing we must do is compute the interest rate for the same period of the annuity, i.e., for the month, i_{12}.

On question i) the interest rate is i = 0,12 (annual effective).

So, i_{12} is computed using an equivalence relationship:

$$(1+i_{12})^{12} = (1+0,12)^1 \Leftrightarrow i_{12} = 0,009489$$

From now on, everything is similar to the previous example, where we had a simple annuity.

a) Moment 8 is a particular moment: it is the moment when the last payment occurs. So, it's very easy to compute the value at that moment: it is exactly FV_A.

$$\mathbf{V_8 = FV_A = 20.000\ s_{\overline{7}|0,009489} = €144.048,92}$$

b) To compute the value of the annuity at moment 10 we can simply compound V_8 (FV_A) two months:

$$\mathbf{V_{10} = V_8\,(1+i_{12})^2 = FV_A\,(1+i_{12})^2 = 20.000\ s_{\overline{7}|0,009489}\,(1+0,009489)^2}$$

$$\mathbf{V_{10} = €146.795,65}$$

c) Moment 1 is a particular moment: it's the origin of this annuity; so, we simply want PV_A:

$$\mathbf{V_1 = PV_A = 20.000\ a_{\overline{7}|0,009489} = €134.834,02}$$

d) To compute the value of the annuity at moment 0 we can simply discount V_1 one month:

$$\mathbf{V_0 = V_1\,(1+0,009489)^{-1} = PV_A\,(1+i_{12})^{-1}}$$
$$\mathbf{V_0 = 20.000\ a_{\overline{7}|0,009489}\,(1+0,009489)^{-1}}$$
$$\mathbf{V_0 = €133.566,61}$$

We could also, for instance, discount V_8 eight months:

$$\mathbf{V_0 = V_8\,(1+0,009489)^{-8} = 144\ 048,92\,(1+0,009489)^{-8}}$$
$$\mathbf{V_0 = €133.566,42}$$

e) To compute the value of the annuity at moment 2 we can simply compound V_1 one month:

$$\mathbf{V_2 = V_1\,(1+i)^1 = PV_A\,(1+0,009489)^1}$$
$$\mathbf{V_2 = 20.000\ a_{\overline{7}|0,009489}\,(1+0,009489)^1}$$
$$\mathbf{V_2 = €136.113,46}$$

On question ii) the interest rate is $i_{(12)}$ = 0,12 (annual nominal, monthly compounded). So, i_{12} is computed using a proportional relationship:

$$i_{12} = 0,12/12 = 0,01$$

From now on, everything is similar to previous examples.

a) Moment 8 is a particular moment: it is the moment when the last payment occurs. So, it's very easy to compute the value at that moment: it is exactly FV_A.

$$\mathbf{V_8 = FV_A = 20.000\ s_{\overline{7}|0,01} = €144.270,70}$$

b) To compute the value of the annuity at moment 10 we can simply compound V_8 (FV_A) two months:

$$\mathbf{V_{10} = V_8\ (1+i_{12})^2 = FV_A\ (1+i_{12})^2}$$
$$\mathbf{V_{10} = 20.000\ s_{\overline{7}|0,01}\ (1+0,01)^2}$$
$$\mathbf{V_{10} = €147.170,55}$$

c) Moment 1 is the origin of this annuity; so, we simply want PV_A:

$$\mathbf{V_1 = PV_A = 20.000\ a_{\overline{7}|0,01} = €134.563,89}$$

d) To compute the value of the annuity at moment 0 we can simply discount V_1 one month:

$$\mathbf{V_0 = V_1\ (1+0,01)^{-1} = PV_A\ (1+i_{12})^{-1}}$$
$$\mathbf{V_0 = 20.000\ a_{\overline{7}|0,01}\ (1+0,01)^{-1}}$$
$$\mathbf{V_0 = €133.231,57}$$

We could also, for instance, discount V_8 eight months:

$$\mathbf{V_0 = V_8\ (1+0,01)^{-8} = 144\ 270,70\ (1+0,01)^{-8} = €133.231,57}$$

e) To compute the value of the annuity at moment 2 we can simply compound V_1 one month:

$$\mathbf{V_2 = V_1\ (1+i)^1 = PV_A\ (1+0,01)^1 = 20.000\ a_{\overline{7}|0,01}\ (1+0,01)}$$
$$\mathbf{V_2 = €135.909,53}$$

Let's now see two examples to show how to solve when the unknown is *n* and the result is a non-integer (Example 14) and when the unknown is *i* (Example 15).

<u>Example 14</u>

Imagine you can save $150 every month. How many deposits must you make in order to accumulate $10,000 if the interest rate is 6%, annual nominal?

<u>Explanation</u>

First, notice that we need the monthly interest rate, because the period of the annuity is the month. To do it, we use a proportional relationship, because the interest rate is nominal. So,

$$i_{12} = 0.06/12 = 0.005$$

Now, we have

$$FV_A = \$10,000$$
$$PMT = \$150$$
$$i_{12} = 0.005$$
$$n = ?$$

Let's solve:

$$10{,}000 = 150\, s_{\overline{n|}0.005}$$

$$10{,}000 = 150\frac{(1+0.005)^n - 1}{0.005}$$

$$1.333333 = 1.005^n$$

$$\ln 1.333333 = n \ \ln 1.005$$

$$n = 57.6801$$

Conclusion: it is not possible to accumulate exactly \$10,000 with an integer number of monthly deposits of \$150 if the interest rate is 0.5% per month. If we make 57 deposits it will not be enough; if we make 58 deposits, FV will be higher than \$10,000.

Now, to accumulate exactly \$10,000, we can, for instance,

1. *Make 58 deposits, 57 of which of \$150 and a final one that will be, for sure, lower than \$150 (say, x); or*

2. *Make 57 deposits, 56 of which of \$150 and a final one that will be, for sure, higher than \$150 (say, y).*

How can we find exactly how much should be x (situation 1) or y (situation 2)? We just build the correct equations:

<u>Situation 1</u>: $10{,}000 = 150\, s_{\overline{57|}0.005}\,(1+0.005)^1 + x$

$$10{,}000 = 150\frac{(1+0.005)^{57} - 1}{0.005}(1+0.005)^1 + x$$

$$x = \$86.14$$

Notice that the last (58^{th}) deposit, x, will take place one month after the last (57^{th}) deposit of \$150 and \$10,000 will be achieved exactly when that deposit, x, is made.

<u>Situation 2</u>: $10{,}000 = 150\, s_{\overline{56|}0.005}\,(1+0.005)^1 + y$

$$y = \$285.46$$

Notice that the last (57^{th}) deposit, y, is made one month after the last (56^{th}) deposit of \$150 and \$10,000 will be achieved exactly when that deposit, y, is made.

And what if the unknown is the interest rate, *i*?

<u>Example 15</u>

Imagine you've been proposed a financial product like this:

. You deposit \$100 every month for 48 months;

. Then, at that exact moment (immediately after you make the last deposit) you receive \$5.000

How much is the annual effective interest rate?

<u>*Explanation*</u>

First of all, notice that we must compute i_{12}, i. e., the interest rate for the month, because the period of the annuity is the month.

The equation to compute i_{12} is

$$5{,}000 = 100\ s_{\overline{48}|i12}$$

$$5{,}000 = 100\frac{\left(1+i_{12}\right)^{48}-1}{i_{12}}$$

It can't be solved analytically. Luckily, this is very easy to solve using a financial calculator or a spreadsheet. The result is i_{12} = 0.00172645, i.e., 0.1726% (per month). So, the annual effective interest rate will be I, such as

$$(1+i)^{1} = (1+0.00172645)^{12} \Leftrightarrow i = 0{,}020915$$

4.6 - Perpetuities.

Usually, **perpetuity** is defined as being an annuity which never ends. Actually, this is not a good definition. It is possible that a finite annuity (i.e., with a certain and even low number of payments) is a perpetuity. What really determines if an annuity is a perpetuity or not is not only the number of cash flows itself (alone); it's the number of cash flows **and** the interest rate, simultaneously. Interest rate plays a very important role here.

Some real life examples of perpetuities may be found on stock (equity) valuation, firm valuation, scholarships paid perpetually from an endowment, insurance and so on.

On a perpetuity we only want to find its present value, which is finite because cash-flows that occur far from now have a low present value (lower and lower as they occur later and later, even reaching zero). Computing future value on a perpetuity would be nonsense (it would be… infinite).

How can we compute the present value of an annuity? Well, we can assume that $n \to \infty$. So, it will be

$$PV_{A(\infty)} = PMT\frac{1-\left(1+i\right)^{-\infty}}{i} = PMT\frac{1}{i}$$

$$\boxed{PV_{A(\infty)} = \frac{PMT}{i}}$$

Two important notes about this (actually, it's nothing new; it was the same on annuities, as we saw previously):

1. This present value, $PV_{A(\infty)}$, as computed, is at the origin of the perpetuity; and
2. The interest rate, i, must be referred to the same period of the perpetuity.

Example 16

A person intends to donate a perpetual scholarship to the best student of Instituto Politécnico de Viseu (IPV). She wants the scholarship to be the amount of €1.000, paid every year, starting one year from now. If the interest rate is 5% (annual, effective) how much must she donate today so that her wish is possible?

Explanation

We can simply say that this lady must donate today the present value of the perpetuity, which means that this amount has to be

$$PV_{A(\infty)} = \frac{PMT}{i} = \frac{1.000}{0,05} = 20.000$$

But let's try to understand what's really happening: this person puts her money (€20.000) on the bank. The interest rate the bank pays is 5% (annual, effective). So, the annual interest is €1.000. Instead of receiving this money, the person gives it to the best student of IPV, as a scholarship. And this happens year after year, forever.

She will never take the €20.000 back from the bank. Indeed, this is no big deal: at 5% per year, in 50 years, having €20.000 euro is worth only the equivalent to €1.744 today:

$$20.000(1+0,05)^{-50} = €1.744$$

And 100 years from now it will be

$$20.000(1+0,05)^{-100} = €152 \ldots$$

Note that if the interest rate was 10%,

1. *This person would need to donate today only €10.000* $\left[PV_{A(\infty)} = \frac{1.000}{0,10} = €10.000 \right]$*; and*

2. *In 50 years, having €10.000 is worth only the equivalent to €85 today…*

$$10.000(1+0,10)^{-50} = €85$$

Exercises

Annuities

4.1 - Consider the following annuity and assuming the annual interest rate of 10% compute its value at the following moments (values in €):

i) Moment 0
ii) Moment 5
iii) Moment 1
iv) Moment 8

Imagine a real life situation that could represent each situation i) to iv).

	1 000	1 000	1 000	1 000	1 000			
0	1	2	3	4	5	6	7	8 (years)

(i: V_0 = €3.790,79; V_5 = €6.105,10; V_1 = €4.169,87; V_8 = €8.125,89)

4.2 - Present value of an annuity with eighteen annual constant and ordinary payments is \$30,000 at the annual interest rate of 12%. Compute the payment. Imagine a real life situation that could represent this situation.

(PMT = \$4,138.12)

4.3 - Future value of an annuity with annual constant and ordinary payments is \$100,000. Knowing that the annual interest rate is 8% and that each payment has the value of \$3,297.69, compute the number of payments of this annuity. Imagine a real life situation that could represent this situation.

(n = 16)

4.4 - Ten annual deposits of €10.000 produce the future value of €158.000. Compute the annual interest rate. Imagine a real life situation that could represent this situation.

(i = 9,819%, using a financial calculator or a spreadsheet)

4.5 - Agata must pay the following amounts to a creditor:

- Five annual payments of €1.000 each;
- Then, five annual payments of €2.000 each.

i) Compute the present value of this set of payments at the annual interest rate of 11,5%. What could this be in real life? *(PV_A = €7.885,67)*

ii) Compute the future value of this set of payments at the annual interest rate of 11,5%. What could this be in real life? *(FV_A = €23.420,02)*

4.6 - Compute the future value and the present value of ten annual payments of \$10.000 each, knowing that until the end of the fourth year the annual interest rate is 9% and from that moment on the annual interest rate is 10%.

(FV_A = \$158,171.87; PV_A = \$63,250.96)

4.7 - Compute the future value and the present value of 44 quarterly payments of €200 each, considering:

i) An effective annual interest rate of 9,5%.
ii) A nominal annual interest rate of 9,5% quarterly compounded.
iii) A nominal annual interest rate of 9,5% compounded every semester.
iv) A nominal annual interest rate of 9,5% monthly compounded.

(i: FV_A = €14.935,19; PV_A = €5.503,71; ii: FV_A = €15.232,62; PV_A = €5.423,04; iii: FV_A = €15.129,65; PV_A = €5.450,55; iv: FV_A = €15.303,56; PV_A = €5.404,33)

4.8 - Jerina made six annual deposits, each one of them of €5.000, in a bank that offered an effective annual interest rate of 7%. Then, she decided to receive all the money she has been accumulating in ten constant semi-annual withdrawals. The first one occurred two years after the sixth annual deposit (it was used the same effective annual rate, 7%). Compute the value of each semi-annual withdrawal.

(PMT = €4.745,79)

4.9 - Dagmara intends to replace an annuity of ten annual payments of $1,000 each, the first one occurring only two years from now, by an annuity due with twenty semi-annual constant payments. Compute the value of each semi-annual payment, assuming the effective annual rate of 8.5%.

(PMT = $433.39)

4.10 - Austéja took a loan of $100,000. This loan must be paid through quarterly payments of $5,250 each, as much as possible. The first payment is to be made one year later. The interest rate is 12%, annual nominal, monthly compounded.

i) Prove that it is not possible to refund correctly the loan with all payments of $5,250.
ii) Adjust the last payment, x, so that, being less than but as close as possible to $5,250, the loan be perfectly refunded.
iii) Adjust the last payment, y, so that, being higher than but as close as possible to $5,250, the loan be perfectly refunded.

(i: n = 33,419...; not an integer, so, not possible; ii: x = $2,219.06; iii: y = $7,403.80)

4.11 - Smokers who want to stop smoking may try a medical treatment that costs €500. They can pay that money in 12 monthly payments of €42,93 each, at the end of each month.

i) What is the annual effective interest rate they pay?
(Note: if you can't solve, at least write the correct equation)
ii) Suppose that António stopped smoking. So, he will save €95 on cigarettes every month. Besides, he is so happy that he decided to reduce beer drinking, which means that he will save another €55 every month. If he puts that money (€150) on a bank at the end of each month at the rate of 5% (annual, effective), how much will he have 10 years later?

(i: i = 5,694%; ii: FV_A = €23.154,47)

4.12 - A rich and altruistic person intends to donate a perpetual scholarship to the best student of Instituto Politécnico de Viseu (IPV). He wants the scholarship to be the amount of €1.000, paid every semester, starting today. If the interest rate is 5,0625% (annual, effective) how much must he donate today so that his wish is possible?

(V_0 = €41.000)

4.13 – Emilia must pay the following amounts (in €):

100	120	100	100	200	100	100	120	100				
0	1	2	3	4	5	6	7	8	9	10	11	12

(months)

Assuming the interest rate of 10%, annual, nominal, quarterly compounded and using concepts of Chapter 4, as much as possible (i.e., not as you should do in Chapter 3, compounding or discounting each cash-flow, one by one),

i) Write the equations to compute the value of all these payments at moment 0 and at moment 12 and compute those values. *(V_0 = €1.006,53)*

ii) Imagine and describe clearly possible (real life) situations that could represent each of those values. *(V_{12} = €1.105,57)*

4.14 - A computer store allows customers to pay their debts in one of three ways:

i) Cash (i.e., when they buy); or

ii) 30% at the moment they buy, plus 12 monthly constant payments, starting one month later; or

iii) 20% at the moment they buy, plus 4 quarterly constant payments, starting three months later.

Diana just bought a computer that costs $1,000. How much will she pay every month if she chooses to pay according to option ii) and every quarter if she chooses to pay according to option iii)? Assume that the interest rate is 16%, annual effective.

(PMT = $63.16 per month or PMT = $219.25 per quarter)

4.15 - Despoina intends to save some money every year until she has accumulated $60,000. She thinks she can save $5,000 per year, starting right now. The bank pays her 5%, annual effective. Can Despoina exactly achieve her goal under these circumstances? If no, compute the amount of the last deposit in order to be higher than the others, but in such a way that she makes as much deposits of $5,000 each as possible.

(Despoina must make 8 deposits of $5,000 each, plus a 9th deposit of $9,867.18)

4.16 - Take the situation of the previous question. Imagine that Despoina really saved the money she wanted, the way she wanted to ($60,000). Then she intends to take that money back from the bank through 36 monthly constant withdrawals, starting 3 months after the last deposit she made. Assuming the same interest rate, how much will she receive per month?

(PMT = $1,809.92)

4.17 - Imagine the following set of payments (in €):

50	50	100	100	150	100	100	200	100	100	150		
0	1	2	3	4	5	6	7	8	9	10	11	12 (months)

Assuming the interest rate of 9%, annual, nominal, quarterly compounded and using concepts of Chapter 4, as much as possible (i.e., not as you should do in Chapter 3, compounding or discounting each cash-flow, one by one), write the equations to compute the value of all these payments at moment 0 and at moment 12 and compute those values.

(V_0 = €1.150,51; V_{12} = €1.257,60)

5. LOAN AMORTIZATION

5.1 - Basic concepts and key variables.

A loan can be amortized in several ways. Indeed, only one thing must be respected: financial equivalence, at any moment, between borrowed amounts (inflows, i.e., positive flows on the perspective of the borrower) and paid amounts (outflows, i.e., negative flows on the perspective of the borrower).

However, some amortization systems are more used than others. We will briefly see two of them usually known as “French System” (constant payments) and “Italian System” (constant amortizations or principal).

First of all, let’s introduce some concepts and variables. These concepts and variables will be used, no matter the system.

<u>**Some important concepts and notation**</u>:

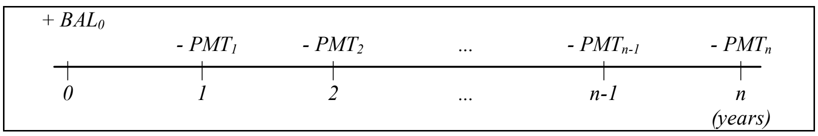
- **Loan amount** or **Initial debt (D_0 or PV)**: borrowed amount.
- **Debt** (or **balance)** at a given moment, k, or after a given payment, k **(D_k or BAL_k)**: amount of the loan still to pay at a given moment, k, or after a given payment, k;
- **Payment** number k **(p_k or PMT_k)**: usually includes two items: interest and amortization (or principal). Only principal reduces the debt.
- **Interest** included in payment k **(I_k or INT_k)**: part of the payment computed upon the remaining debt at the end of period (k-1). It does not reduce the debt. $I_k = D_{k-1}.i$ (or $INT_k = BAL_{k-1}.i$)
- **Amortization** or **principal** included in payment k **(m_k or PRN_k)**: part of the payment that is subtracted to the debt.

  $m_k = p_k - I_k$ (or $PRN_k = PMT_k - INT_k$)

  So, $p_k = m_k + I_k$ (or $PMT_k = PRN_k + INT_k$).

- **Interest rate (i):** must be expressed at the same period of payments i.e., if payments are to be made every month, we must use the monthly interest rate, correctly computed (as seen on Chapter 2)
- **Grace period**: period during which the borrower pays no principal (i.e., he or she only pays interest) or even nothing at all (in this case, the debt will be increased by interest, under compound interest).

Let’s assume that the loan will be paid this way:



This means that, for now, we assume that the payments are made every year, the first one occurring one year after the loan was borrowed. Later we will drop these assumptions.

5.2 - Amortization schedule.

The amortization schedule is a very used tool when dealing with loans. It may have several aspects, but the main information it contains is this one:

AMORTIZATION SCHEDULE

Period k or **Payment k**	**Debt at the beginning of period k (i.e., before payment k) (Balance) (D_{k-1} or BAL_{k-1})**	**Interest in payment k (INT_k)**	**Amortization (principal) in payment k (m_k or PRN_k)**	**Payment k (p_k or PMT_k)**
1	BAL_0	$INT_1=BAL_0.i$	PRN_1	$PMT_1=INT_1+PRN_1$
2	$BAL_1=BAL_0-PRN_1$	$INT_2=BAL_1.i$	PRN_2	$PMT_2=INT_2+PRN_2$
...	...	...	...	...
n-1	$BAL_{n-2}=BAL_{n-3}-PRN_{n-2}$	$INT_{n-1}=BAL_{n-2}.i$	PRN_{n-1}	$PMT_{n-1}=INT_{n-1}+PRN_{n-1}$
n	$BAL_{n-1}=BAL_{n-2}-PRN_{n-1}$	$INT_n=BAL_{n-1}.i$	PRN_n	$PMT_n=INT_n+PRN_n$

1. These two amounts must be equal

Σ PRN=BAL_0

2. The sum of all amortizations has to be equal to the amount of the loan

Two notes about the amortization schedule (no matter the amortization system):

1. **BAL_{n-1} = PRN_n** (n: total number of payments)
   In fact, if $BAL_{n-1} > PRN_n$, that would mean that the loan was not totally paid yet and if $BAL_{n-1} < PRN_n$, that would mean that the loan was overpaid.

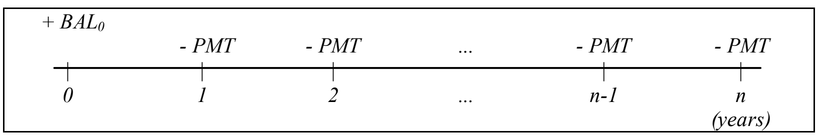
2. **Σ PRN = BAL_0**
   In fact, only amortizations (PRN_k) pay the debt. So, the sum of all amortizations must equal the amount of the loan.

5.3 - Two systems to amortize debt loans.

As we told before, debt can be amortized in many ways. Next, we will see two of them.

5.3.1 - French System (constant payments)

In this system of amortization, all payments are the same amount. So, it is something like this:



As we can see, this is just like annuities, on Chapter 4. Pay attention to the similarities between BAL_0 and PV_A. In fact, BAL is exactly the present value of all the *n* payments (PMT). So, we can say that $\boxed{BAL_0 = PMT.a_{\overline{n|}\,i}}$. In fact, as we saw on Chapter 4, $\boxed{PV_A = PMT.a_{\overline{n|}\,i}}$ (see page 27).

Example 17

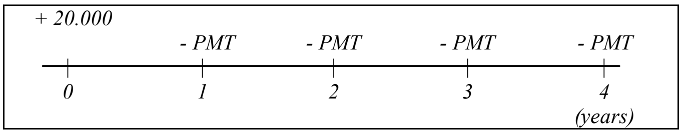
A person borrowed this loan:
- Amount: €20.000
- Time: 4 years
- Interest rate: 6%, annual effective
- Payments: constant, annual, starting one year after the loan has been borrowed

We want to
1. *Compute the amount of each payment*
2. *Build the amortization schedule*

Explanation

1. *This is what we have:*
   $BAL_0 = €20.000$
   $n = 4$ *(annual constant payments)*
   $i = 6\%$ *(annual effective)*
   $PMT = ?$

In a diagram, it is something like this:



So,

$$\underbrace{20.000}_{\substack{\text{Amount of the loan} \\ \text{(moment 0)}}} = \underbrace{PMT.a_{\overline{4|}\,0,06}}_{\substack{\text{Value of the 4 payments} \\ \text{at moment 0}}}$$

$PMT = €5.771,83$

2. *Amortization schedule:*

Period k or Payment k	Debt at the beginning of period k (i.e., before payment k) (Balance) (D_{k-1} or BAL_{k-1})	Interest in payment k (INT_k)	Amortization (principal) in payment k (m_k or PRN_k)	Payment k (p_k or PMT_k)
1	20.000,00	1.200,00	4.571,83	5.771,83
2	15.428,17	925,69	4.846,14	5.771,83
3	10.582,03	634,92	5.136,91	5.771,83
4	5.445,12	326,71	5.445,12	5.771,83

Explanation about building the amortization schedule:
1. *Compute PMT:* $20.000 = PMT.a_{\overline{4|}\,0,06} \Leftrightarrow PMT = €5.771,83$
2. *Compute* INT_1: $INT_1 = BAL_0.i = 20.000 \times 0,06 = €1.200$
3. *Compute* PRN_1: $PRN_1 = PMT - INT_1 = 5.771,83 - 1.200 = €4.571,83$
4. *Compute* $BAL_1 = BAL_0 - PRN_1 = 20.000 - 4.571,83 = €15.428,17$

And so on. Notice that, as we told on page 41,
1. $BAL_3 = PRN_4$ *(5.445,12 >> last line of the Amortization Schedule); and*
2. *The sum of all amortizations* $\Sigma\, PRN = BAL_0$.

There are **some interesting things** about French System, related to

1. How amortizations grow from one period to the next one (Amortization's Law)
2. First amortization (or principal, PRN_1)
3. Last amortization (or principal, PRN_n)

- **Amortizations' Law**

In any payment the debtor amortizes part of the debt; so, as time goes by, the debt gets lower and lower; because of this, the interest in any payment will also be lower; as the payment remains constant, the amortization (principal) will increase from one payment to the next one. This is easy to understand. The interesting thing is that under French System the amortizations grow in a very particular way: they follow a geometric progression, the multiplying constant being (1+i). This is,

$$\underbrace{PRN_2}_{4.846,14} = \underbrace{\underbrace{PRN_1}_{4.571,83}(1+0,06)}_{4.846,14}\ ; \qquad \underbrace{PRN_3}_{5.136,91} = \underbrace{\underbrace{PRN_2}_{4.846,14}(1+0,06)}_{5.136,91}\ ; \qquad \text{... etc. ...}$$

Indeed, we can prove that $\boxed{PRN_{k+1} = PRN_k(1+i)}$ for any k.

So,

$$\boxed{PRN_k = PRN_1(1+i)^{k-1}}$$

In general,

$$\boxed{PRN_k = PRN_j(1+i)^{k-j}}$$

In our example, as we saw previously, PRN_1 = 4.571,83. So, according to Amortizations' Law, we could say, for instance, that

$$PRN_3 = PRN_1\ (1+i)^2$$
$$PRN_3 = 4.571,83\ (1+0,06)^2$$
$$PRN_3 = 5.136,91 \quad \text{(check the Amortization Schedule)}$$

- **First amortization (principal)**

We know that

$$BAL_0 = PRN_1 + PRN_2 + ... + PRN_{n-1} + PRN_n$$

and that

$$PRN_2 = PRN_1\ (1+i)$$
$$PRN_3 = PRN_2\ (1+i) = PRN_1\ (1+i)^2$$
$$PRN_4 = PRN_3\ (1+i) = PRN_1\ (1+i)^3$$
$$\text{... etc. ...}$$
$$PRN_n = PRN_{n-1}\ (1+i) = PRN_1\ (1+i)^{n-1}$$

So,

$$BAL_0 = PRN_1 + PRN_1(1+i) + PRN_1(1+i)^2 + ... + PRN_1(1+i)^{n-1}$$

This means that BAL_0 is the sum of the terms of a geometric progression with these characteristics:

. First term (value) of the progression (t_1) = PRN_1
. Number of terms of the progression (n) = n
. Multiplying constant (c) = (1+i)

From Mathematics, we know that the sum of all the n values of a geometric progression, S_{GP}, is

$$\boxed{S_{GP} = t_1 \frac{c^n - 1}{c - 1}}$$

Well, in this case we have the following correspondence:

Geometric Progression		French System
S_{GP}	⇔	BAL_0
t_1	⇔	PRN_1
c	⇔	(1+i)
n	⇔	n

So, we can say that under French System

$$BAL_0 = PRN_1 \frac{(1+i)^n - 1}{(1+i) - 1}$$

$$BAL_0 = PRN_1 \underbrace{\frac{(1+i)^n - 1}{i}}_{s_{\overline{n}|i}}$$

$$\boxed{BAL_0 = PRN_1 . s_{\overline{n}|i}}$$

Or, which is the same, $\boxed{PRN_1 = \frac{BAL_0}{s_{\overline{n}|i}}}$

In our example:

$$PRN_1 = \frac{20.000}{\frac{(1+0,06)^4 - 1}{0,06}}$$

$$PRN_1 = \frac{20.000}{4,374616}$$

$$PRN_1 = 4.571,83 \text{ (check the Amortization Schedule)}$$

- **Last amortization (principal)**

We know that $PMT_n = PRN_n + INT_n$ which is the same as

$$PMT_n = PRN_n + \underbrace{BAL_{n-1}.i}_{INT_n} \qquad \textbf{[1]}$$

On the other hand, the last amortization must equal the debt at the beginning of the last period. So,

$$BAL_{n-1} = PRN_n$$

Replacing in [1], we get

$$\underbrace{PMT_n}_{\text{Last payment}} = \underbrace{PRN_n}_{\text{Last amortization}} + \underbrace{\underbrace{PRN_n}_{\text{Debt at the beginning of the last period}} \quad . \, i}_{\text{Last interest}}$$

So,

$$\boxed{PMT_n = PRN_n(1+i)}$$

Or

$$PRN_n = PMT_n(1+i)^{-1}$$

It happens that under French System $PMT_1 = PMT_2 = PMT_3 = ... = PMT_n$. So, we can say that

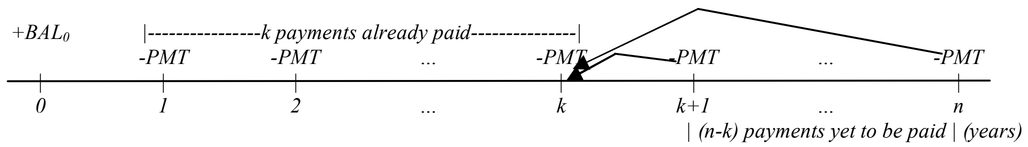
$$\boxed{PRN_n = PMT\,(1+i)^{-1}}$$

In our example we have 4 payments (n=4). So, $PRN_4 = 5.771{,}83\ (1+0{,}06)^{-1}$, i.e., $PRN_4 = 5.445{,}12$ (check the Amortization Schedule).

- **Debt at any moment**

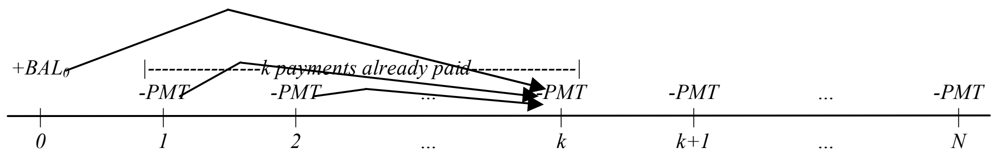
To compute the debt after payment k, BAL_k, we can do one of the following:

1. Discount to that moment every payment not paid yet (i.e., (n-k) payments:



This is BAL_k, obtained by $BAL_k = PMT.a_{\overline{n-k|}i}$ (value of the (n-k) payments not paid yet, at moment k)

2. Compound to that moment the amount of the initial debt and then subtract the amount of every payment already paid, i.e., k payments, referred to that moment:



This is BAL_k, obtained by $BAL_k = BAL_0(1+i)^k - PMT.s_{\overline{k|}i}$ (value of the difference between

a) The initial debt, compounded to moment k; and
b) All payments already paid (k payments), compounded to moment k

In our example we could compute the debt at moment 2 (i.e., immediately after payment 2) this way

1.

$$BAL_2 = \underbrace{5.771,83\frac{1-(1+0,06)^{-2}}{0,06}}_{\text{value, at moment 2, of the 2 remaining payments, i.e., not paid yet}}$$

$$BAL_2 = 10.582,03 \text{ (check the Amortization Schedule)}$$

or this way

2.

$$BAL_2 = \underbrace{20.000(1+0,06)^2}_{\text{value of the debt at moment 2 (as if nothing had been paid yet)}} \underbrace{-5.571,83\frac{(1+0,06)^2-1}{0,06}}_{\text{value, at moment 2, of the 2 payments already paid}}$$

$$BAL_2 = 22.472 - 11.889,97$$

$$BAL_2 = 10.582,03 \text{ (check the Amortization Schedule)}$$

Soon we will drop the assumptions admitted until now: we will see what to do if payments are not annual and/or the first payment doesn't occur at the end of the first period.

- **Total interest**

How can we compute total interest in such a loan? Well, it will be simply the difference between total payments and the amount of the loan (or, as we saw, Σ PRN). So,

Σ INT = Σ PMT - Σ PRN

Now: let's drop the assumptions admitted until now. Let's see what to do if payments are not annual and/or the first payment doesn't occur at the end of the first period.

Payments not annual (or, more generally, payments frequency different from interest rate period)

What if the period of the interest rate and the period of the payments are different? No problem. We simply must compute the interest rate to the same period of the payments (remember what we saw on Chapter 2, section 2.2).

<u>**Example 18**</u>

Imagine a loan of €200.000 to be paid in 180 monthly and constant payments, starting one month later, at the annual effective interest rate of 14%.
Show the first three and the last lines of the amortization schedule.

<u>*Explanation*</u>

This is what we have:

+200.000	-PMT	-PMT	-PMT	...	-PMT	-PMT
0	1	2	3	...	179	180 (months)

The payments are made every month, but the interest rate is annual. So, we must start by computing the monthly interest rate. Because the given rate is effective, we must use an equivalence relationship.

$$(1+i_{12})^{12} = (1+0,14)^{1}$$
$$i_{12} = (1,14)^{1/12} - 1$$
$$i_{12} = 0,010979 \text{ (actually, it is 0,010978852)}$$

So,

$$200.000 = PMT.a_{\overline{180}|\,0,010979}$$
$$200.000 = PMT.\frac{1-1,010979^{-180}}{0,010979}$$
$$200.000 = 78,323628PMT$$
$$PMT = €2.553,51$$

AMORTIZATION SCHEDULE

PMT nr	Debt at the beginning of period k (BAL_{k-1})	Interest in payment k (INT_k)	Amortization (principal) in payment k (PRN_k)	Payment k (PMT_k)
1	200.000,00	2.195,77	357,74	2.553,51
2	199.642,26	2.191,84	361,67	2.553,51
3	199.280,59	2.187,87	365,64	2.553,51
…	…	…	…	…
180	2.525,78	27,73	2.525,78	2.553,51

Explanation:

1. Compute PMT: $PMT = €2.553,51$ (as we just saw)
2. Compute INT_1: $INT_1 = BAL_0.i_{12} = 200.000x0,010979 = €2.195,77$*
* We took $i_{12} = 0,010978852$
3. Compute PRN_1: $PRN_1 = PMT - INT_1 = 2.553,51 - 2\ 195,77 = €357,74$
4. Compute $BAL_1 = BAL_0 - PRN_1 = 200.000 - 357,74 = €199.642,26$

And so on.

And how to compute the values of the last line without having to compute all previous values? One easy way could be this one:

a) $PRN_{180} = PRN_1\ (1+i_{12})^{179}$
$PRN_{180} = 357,74\ (1+0,010978852)^{179}$
$PRN_{180} = €2.525,79$
b) $BAL_{179} = PRN_{180}$
$BAL_{179} = €2.525,79$
c) $INT_{180} = BAL_{179} \times i_{12}$
$INT_{180} = 2.525,79 \times 0,010978852$
$INT_{180} = €27,73$
d) $PMT_{180} = INT_{180} + PRN_{180}$
$PMT_{180} = 27,73 + 2.525,79$
$PMT_{180} = €2.553,52$

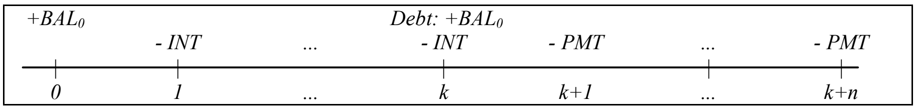
There are other ways to compute the values of the last line of the Amortization Schedule. Can you identify one or two?

First payment not at the end of the first period

. French System with grace period

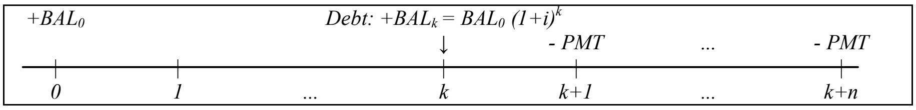
Sometimes there's an initial period during which the debtor does not amortize debt. He or she only pays interest or even nothing. This is usually called "grace period".

1. Grace period with payment of interest (grace period of k periods)



In this situation, the debt at moment k is BAL_0, because in the previous periods the interest has been paid. So, $BAL_0 = PMT.a_{\overline{n|}\,i}$ just like before (the only difference is that, now, the debtor pays the interest during the grace period).

2. Grace period with no payment of interest (grace period of k periods)



In this situation, the debt at moment k is $BAL_k = BAL_0\ (1+i)^k$, because in the previous periods nothing has been paid. From moment k until the end, we have n payments. So, $BAL_0\,(1+i)^k = PMT.a_{\overline{n|}\,i}$. All we are doing is compounding BAL_0 to moment k and then computing BAL_k as a regular annuity.

Example 19

Imagine a loan of \$50,000 to be paid with 20 quarterly and constant payments, after a grace period of one year, at the annual nominal interest rate of 8%, quarterly compounded.

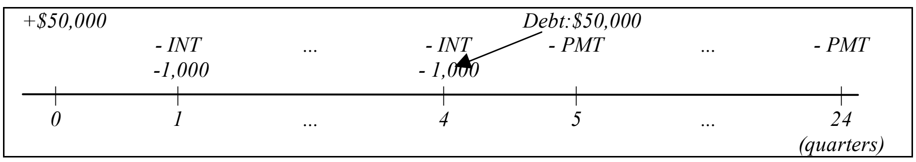
a) Consider that during the grace period the borrower pays interest every quarter. How much will he or she pay quarterly during the first year and after that?

b) Consider now that the borrower pays nothing during the grace period. How much will he or she pay after that?

Explanation

a) In this situation, during the first year the borrower will pay only interest, i.e.

INT = 50,000 x 0.02 = \$1,000 *(i_4 = 0.08/4 = 0.02)*



At the end of the first year the debt still remains at \$50,000. So, to compute the value of each payment from that moment on, we will do this:

$$50{,}000 = PMT.a_{\overline{20|}\,0.02}$$

$$50{,}000 = PMT.\frac{1-(1+0.02)^{-20}}{0.02}$$

$$PMT = \$3{,}057.84$$

b) In this situation, at the end of the first year the debt will be \$50,000 $(1+0.02)^4$ = \$54,121,61. In a diagram, it is

+\$50,000			Debt: \$54,121.61 ↓	- PMT	...	- PMT
		...				
0	1	...	4	5	...	24 (quarters)

So, the value of each payment from moment 5 to moment 24 will be the result of the following equation:

$$54{,}121.61 = PMT.a_{\overline{20|}\,0.02}$$

$$54{,}121.61 = PMT.\frac{1-(1+0.02)^{-20}}{0.02}$$

$$PMT = \$3{,}309.90$$

Leasing

Leasing is, we can say, a special case of French System. There is always a final payment (FP) and it may also happen that there is an initial payment (IP) different of all other payments. This should bring no difficulty. All we must keep in mind is that, as always, equivalence must be respected. Let's see some usual situations:

1. No initial payment; *n* ordinary constant annuities (PMT) plus a final payment (FP)

+BAL_0	- PMT	- PMT	...	- PMT	-FP - PMT
0	1	2	...	n-1	n

The corresponding equation is

$$\boxed{BAL_0 = PMT.a_{\overline{n|}\,i} + FP(1+i)^{-n}}$$

2. No initial payment; *n* constant annuities due (PMT) plus a final payment (FP)

+BAL_0 - PMT	- PMT	- PMT	...	- PMT	-FP
0	1	2	...	n-1	n

The corresponding equation is

$$\boxed{BAL_0 = PMT.a_{\overline{n|}\,i}(1+i) + FP(1+i)^{-n}} \text{ or } \boxed{BAL_0 = PMT + PMT.a_{\overline{n-1|}\,i} + FP(1+i)^{-n}}$$

3. An initial payment (IP), plus *(n-1)* constant ordinary annuities (PMT) and a final payment (FP)

$+BAL_0$ $-IP$	$-PMT$	$-PMT$	...	$-PMT$	$-FP$
0	1	2	...	n-1	n

The corresponding equation is

$$\boxed{BAL_0 = IP + PMT.a_{\overline{n-1|}\,i} + FP(1+i)^{-n}}$$

Bear in mind that none of these are new formulas you need to learn. This is always a use of financial equivalence to discount cash flows to moment 0.

Example 20

Imagine a leasing as follows:
- Loan: €20.000
- Time: 4 years
- Interest rate: 6%, annual nominal, monthly compounded
- Final payment: 2% of the loan

Compute the payment for these scenarios:

1) No initial payment; 48 ordinary annuities plus final payment
2) No initial payment; 48 due annuities plus final payment
3) Initial payment: 20% of the loan; 47 ordinary annuities plus final payment

Explanation

We must start by computing the monthly interest rate. It is $i_{12} = \frac{0,06}{12} = 0,005$.

Besides, final payment is €400 (20.000 x 0,02).

1. This is what we have:

+20.000	- PMT	- PMT	...	- PMT	- 400 - PMT
0	1	2	...	47	48 (months)

So,

$$\underbrace{20.000}_{\text{Loan (mom. 0)}} = \underbrace{PMT.a_{\overline{48|}\,0,005}}_{\text{Value of the 48 payments at mom. 0 (origin)}} + \underbrace{400\left(1+0,005\right)^{-48}}_{\text{Value of the final payment at mom. 0}}$$

PMT = €462,31

2. Now we have this:

0	1	2	...	47	48 (months)
+20.000 - PMT	- PMT	- PMT	...	- PMT	- 400

So,

$$\underbrace{20.000}_{\text{Loan (mom. 0)}} = \underbrace{\underbrace{PMT.a_{\overline{48|}\,0,005}}_{\text{Value of the 48 payments at mom. -1 (origin)}}(1+0,005)^{1}}_{\text{Value at mom. 0}} + \underbrace{400\left(1+0,005\right)^{-48}}_{\text{Value of the final payment at mom. 0}}$$

PMT = €460,01

3. Finally, we have this:

0	1	2	...	47	48 (months)
+20.000 - 4.000	- PMT	- PMT	...	- PMT	- 400

So,

$$20.000 = 4.000 + PMT.\ a_{\overline{47|}0,005} + 400\ (1,005)^{-48}$$

PMT = €375,30

5.3.2 - Italian System (constant amortization)

Under this system, as the name itself shows, it's not the *payment* (PMT) but the *amortization* (principal, PRN) that is constant. We have

0	1	2	...	n-1	n
$+ BAL_0$	$- PMT_1$	$- PMT_2$	...	$- PMT_{n-1}$	$- PMT_n$

Now the payments are not constant, because the amortization is constant and the interest will decrease from one payment to the next one.

Each amortization will be $PRN = \dfrac{BAL_0}{n}$.

- **Debt at any moment**

On this system the debt at any moment, BAL_k, is simply computed by multiplying PRN by the number of payments not paid yet, (n-k):

$$\boxed{BAL_k = (n-k).PRN}$$

- **Interest and payment over time**

Both interest and payment decrease *PRN.i* from one period to the next one. In fact, we amortize *PRN* in every payment; so, in the following payment we will pay less *PRN.i* of interest (and, of course, in payment, because *PRN* is constant).

This means that both interest and payment follow an arithmetic progression over time, the constant being *–PRN.i* ("minus" *PRN.i*).

- **Total interest**

Having in mind what we just said, total interest is the sum of all the terms of an arithmetic progression. From Mathematics we know that this sum is

$$S_{AP} = n\frac{t_1 + t_n}{2}$$

where

S_{AP}: sum of the n terms (values) of the progression
t_1: first term of the progression
t_n: last term of the progression
n: number of terms of the progression

So, in Italian System

$$Total\ interest = n\frac{INT_1 + INT_n}{2}$$

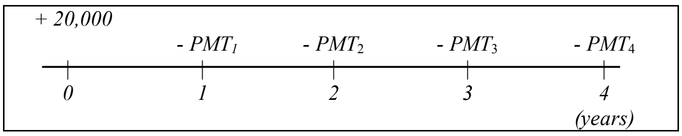
<u>Example 21</u>

A person borrowed this loan:
- Amount: $20,000
- Time: 4 years
- Interest rate: 6%, annual effective
- Payments: annual, starting one year after the loan has been borrowed, where the principal is constant

Build the amortization schedule.

<u>Explanation</u>

We must understand that the payments will not be constant, now. As we know, they contain interest and amortization. In this case, only amortization (principal) is constant. So, this is what we have:



We can easily compute amortization contained in every payment, PRN. In fact,

$$PRN = \frac{BAL_0}{n} = \frac{20,000}{4} = 5,000$$

Let's build the amortization schedule.

PMT nr	Debt at the beginning of period k (BAL_{k-1})	Interest in payment k (INT_k)	Amortization (principal) in payment k (PRN_k)	Payment k (PMT_k)
1	20,000.00	1,200.00	5,000.00	6,200.00
2	15,000.00	900.00	5,000.00	5,900.00
3	10,000.00	600.00	5,000.00	5,600.00
4	5,000.00	300.00	5,000.00	5,300.00

Explanation:

1. Compute PRN: $PRN = \dfrac{BAL_0}{n} = \dfrac{20{,}000}{4} \Leftrightarrow PRN = \$5{,}000$

2. Compute INT_1*:* $INT_1 = BAL_0 \,.\, i = 20{,}000 \text{ x } 0.06 = \$1{,}200$

3. Compute PMT_1*:* $PMT_1 = PRN + INT_1 = 5{,}000 + 1{,}200 = \$6{,}200$

4. Compute $BAL_1 = BAL_0 - PRN = 20{,}000 - 5{,}000 = \$15{,}000$

And so on.

Notice that:

1. $BAL_3 = PRN$ *(5,000.00 >> last line of the Amortization Schedule); and*

2. The sum of all amortizations = $n.PRN = 4 \text{ x } 5{,}000.00 = BAL_0$
This always happens, no matter the amortization system, as we saw (page 41).

Exercises

Loan amortization

5.1 - Tristan took a loan of €120.000, payable through 6 annual, constant and ordinary payments. The annual interest rate is 7% (effective).

i) How much will he pay every year?
ii) Compute the amortization in the first, fourth and last payments
iii) Compute the debt after the fourth payment
iv) Compute the interest in the fifth payment
v) Compute total interest

(i: PMT = €25.175,50; ii: PRN_1 = €16.775,50; PRN_4 = €20.550,70; PRN_6 = €23.528,50; iii: BAL_4 = €45.517,75; iv: INT_5 = €3.186,24; v: Total interest = €31.052,97)

5.2 - Imagine now the previous loan, but with a grace period of two years and then, six annual, constant and ordinary payments. Answer the same questions as before, but assuming now that

a) During the grace period the debtor pays the interest at the end of the first and second years
b) During the grace period the debtor pays nothing

(Question a) i: PMT = €25.175,50; ii: PRN_1 = €16.775,50; PRN_4 = €20.550,70; PRN_6 = €23.528,50; iii: BAL_4 = €45.517,75; iv: INT_5 = €3.186,24; v: Total interest = €47.852,97)
(Question b) i: PMT = €28.823,43; ii: PRN_1 = €19.206,27; PRN_4 = €23 528,50; PRN_6 = €26.937,78; iii: BAL_4 = €52.113,28; iv: INT_5 = €3.647,93; v: Total interest = €52.940,58)

5.3 - Christina has just negotiated this leasing:

. Loan: $25.000
. Time: 4 years
. Monthly payments
. Interest rate: 6% (annual, nominal, monthly compounded)
. Final payment: 4% of the amount of the contract

Compute the following, for situations 1, 2 and 3 as studied before (note: on situation 3 assume that the Initial Payment is 10% of the amount of the loan):

i) Payment
ii) Debt at the end of the first year
iii) Interest and amortization in the 13^{th} payment
iv) New payment if the interest rate changes to 7,2% at the end of the first year

(Situation 1) i: PMT = \$568.64; ii: BAL_{12} = \$19,527.44; iii: INT_{13} = \$97.64; PRN_{13} = \$471.00; iv: PMT' = \$579.77)
(Situation 2) i: PMT = \$565.81; ii: BAL_{13} = \$18,961.58; iii: INT_{13} = \$97.15; PRN_{13} = \$468.66; iv: PMT' = \$576.64)
(Situation 3) i: PMT = \$519.53; ii: BAL_{12} = \$17,479.04; iii: INT_{13} = \$87.40; PRN_{13} = \$432.13; iv: PMT' = \$529.56)

5.4 - Çagla took this loan:

. €100.000
. 5 years
. Quarterly ordinary payments
. Interest rate: 8,2432% (annual effective)
. Italian System

Compute:

i) Amortization in the 10^{th} payment
ii) Debt after 8^{th} payment
iii) Interest in 9^{th} payment
iv) Total interest

(i: PRN = €5.000; ii: BAL_8 = €60.000; iii: INT_9 = €1.200; iv: Total interest = €21.000)

6. BASICS OF INVESTMENT VALUATION.

6.1 - Tangible investments and financial investments.

Tangible investments: investments in buildings, machines, etc.; financial investments: investments in equity (shares), bonds, etc..

6.2 - Evaluation of tangible investments.

Once computed the cash-flows associated to the investment, we can use previous concepts of Time Value of Money to evaluate the opportunity of that investment. The hard work is in predicting those cash-flows. Evaluate them is easy. There are two "classical" indicators: NPV (Net Present Value) and IRR (Internal Rate of Return).

. **NPV** is the present value of the sum of all cash-flows associated to the investment (positive minus negative). If NPV>0, the investment is "interesting"; if NPV<0, the investment is "not interesting".

. **IRR** is the rate that, used to discount every cash-flow, leads to NPV=0, which is, theoretically, the "point of indifference" between going ahead or not with the investment.

Example 22

Suppose that, after a hard study, we hope that a given investment will produce the following cash-flows (in €):

Year	*Investment*	*Cash-Flow*
0	*- 60.000*	
1	*- 20.000*	*10.000*
2		*30.000*
3		*30.000*
4		*20.000*
5		*10.000*

We also believe that this investment will generate a final cash-flow of €2.000 at the end of the 5th year. Assuming the rate of 12% (annual) to evaluate this investment, what should we decide?

Explanation

This is what we have:

					+2.000
-60.000	*-20.000*				
	+10.000	*+30.000*	*+30.000*	*+20.000*	*+10.000*
0	*1*	*2*	*3*	*4*	*5*
					(years)

To evaluate the investment we can compute its NPV (Net Present Value):

$$\underbrace{NPV}_{Mom.\ 0} = \underbrace{-60.000}_{Mom.\ 0} + \underbrace{\underbrace{(10.000-20.000)}_{Mom.\ 1}(1+0,12)^{-1}}_{Mom.\ 0} + \underbrace{\underbrace{30.000}_{Mom.\ 2}(1+0,12)^{-2}}_{Mom.\ 0} +$$

$$+\underbrace{\underbrace{30.000}_{Mom.\ 3}(1+0,12)^{-3}}_{Mom.\ 0} + \underbrace{\underbrace{20.000}_{Mom.\ 4}(1+0,12)^{-4}}_{Mom.\ 0} + \underbrace{\underbrace{(10.000+2.000)}_{Mom.\ 5}(1+0,12)^{-5}}_{Mom.\ 0}$$

$$NPV = €-4.139,86$$

So, the investment seems to be not interesting (NPV<0).

This same conclusion could be obtained using IRR:

$$0 = -60.000 + (10.000 - 20.000)(1+IRR)^{-1} + 30.000(1+IRR)^{-2} +$$

$$+30.000(1+IRR)^{-3} + 20.000(1+IRR)^{-4} + (10.000+2.000)(1+IRR)^{-5}$$

The value of IRR is easily obtained using a financial calculator or a spreadsheet. In this case, IRR = 9,6% (approx.). Being less than 12%, the return is lower than the desired; so, the investment seems to be "not interesting".

6.3 - Evaluation of financial investments.

Equity (shares) is capital of a company. Shareholders are, we can say, owners of the company. They expect two kinds of gains: from dividends, while they keep the shares, and from the sale of the shares. But, of course, both are uncertain. Dividends may or not be distributed to shareholders and even if they are, their amount is uncertain. And, of course, no one can tell, for sure, that the price of the shares will be higher later, when the shareholder decides to sell them. So, it's a risky asset (investment).

Bonds are an instrument of debt through which firms and Governments can borrow money. There are many kinds of bonds. We will only see one type of bonds: Treasury Constant Coupon Bonds. This means that they are a very low risk asset, because the debtor is the Government and because the interest rate ("coupon rate") is constant. In fact, it is usually considered as a riskless asset. The gains the investor expects to earn are interests. The interest (or *coupon*) is computed upon the *face value* of the bond. At the end of the bond's life (*maturity*) the investor receives also the *redemption value* (usually, equal to the face value).

Example 23

Imagine this about ABC shares:

. Today's price: $39.20

. Estimated dividends per share:

. End of year 1: $1.20

. End of year 2: $1.00

. End of year 3: $1.00

. Estimated share price at end of year 3: $45.00

How interesting is the investment in these shares, assuming the given data?

Explanation

This is what we have:

0	1	2	3 (years)
- 39.20	+1.20	+1.00	+ 45.00 +1.00

So, we can compute the annual return, r, solving this equation:

$39.20 = 1.20(1+r)^{-1} + 1.00(1+r)^{-2} + 46.00(1+r)^{-3}$

r=7.3% (computed with a financial calculator or a computer).

Is this interesting? It depends… It depends essentially on two things:

1. *The interest rates on the market, for assets with similar risk. If we can find other assets with similar risk giving higher return, this investment will not be interesting;*

2. *The risk profile of the investor. If the investor has high risk aversion, maybe he or she will find this return too low to compensate the inherent risk.*

Example 24

Imagine this about some Treasury Bonds:

. Price (today): $52.37
. Face value: $50.00
. Redemption value: $50.00
. Coupon rate: 12% (annual)
. Maturity: 5 years
. Return required by the investor: 10%

Should the investor buy these bonds?

Explanation

Let's see: if the investor buys these bonds he or she will pay (today) $52.37 per bond and will receive $6.00 every year (interest = $50.00 x 0.12) plus $50.00 at the maturity (redemption value). So,

0	1	2	3	4	5 (years)
-52.37	+6.00	+6.00	+6.00	+6.00	+50.00 +6.00

Thus, these bonds are worth for the investor, today,

$$\underbrace{Value}_{Mom.\ 0} = \underbrace{6.00a_{\overline{5|}\,0.10}}_{Mom.\ 0} + \underbrace{\underbrace{50.00}_{Mom.\ 5}(1+0.10)^{-5}}_{Mom.\ 0}$$

$Value\ today = \$53.79$

Today's market price is $52.37; so, the investor will tend to buy these bonds, because according to his or her evaluation, they are "cheap" (cost: $52.37; value: $53.79).

Another way of evaluating this investment could be finding the return rate (more or less like IRR, but in bonds this is usually called YTM – Yield to Maturity).

$$\underbrace{52.37}_{\text{Price at mom. 0}} = \underbrace{6.00a_{\overline{5|}\,YTM}}_{\text{Mom. 0}} + \underbrace{\underbrace{50.00}_{\text{Mom. 5}}(1+YTM)^{-5}}_{\text{Mom. 0}}$$

Using a financial calculator or a computer,

YTM = 10.73% (>10%, so the investor finds it a good investment).

Exercises

Tangible investments

6.1 - It is expected that a given tangible investment will produce the following cash-flows (in €):

Year	Investment	Cash-Flow
0	- 500.000	
1	- 120.000	80.000
2		120.000
3		300.000
4		250.000
5		100.000

It is also expected that this investment will generate a residual cash-flow of €70.000 at the end of 5th year. Assuming the rate of 15% (annual) how do you evaluate this investment? Why? What if the rate to evaluate the investment was 12% (annual)?

(At 15%, not interesting: NPV = €-19.332; IRR = 13,75%; at 12%, interesting: NPV = €28.825)

Equity

6.2 - Mantas is studying a possible investment in ABC shares, like this:

. Today's price: €3,35
. Estimated dividends per share:
. End of year 1: €0,20
. End of year 2: €0,22
. End of year 3: €0,23
. Estimated share price at end of year 3: €4,15

If Mantas requires a return of 12% (annual) should he or she buy these shares?

(Yes, because for him these bonds are worth €3,46 today and they cost only €3,35; or using IRR: IRR = 13,41%, which is higher than Mantas requires, 12%, so he will decide to buy)

Bonds

6.3 - Aneta is wondering if she must invest in these Treasury Bonds:

. Price (today): €54,11
. Face value: €50,00
. Redemption value: €50,00
. Coupon rate: 8% (annual)
. Maturity: 4 years
. Return required by the investor: 6%

Should Aneta buy these bonds?

(No, because for her these bonds are worth €53,47 today and they cost €54,11; or using IRR: IRR = 5,65%, which is lower than Aneta requires, 6%, so she will decide not to buy)